Representations of Quivers via Double Affine Hecke Algebras

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The Literature

Of course, there are many sources that one can use. However, the following were those which acted as my bible for the survey of the areas that were relevant to my research.

- [CB99] is used to study representations of quivers.
- [Obl03, Sto05, Mac03] is used to introduce double affine Hecke algebras.
- [CBS04, EOR06] is used to study the Deligne-Simpson problem.
- [CM10] is used to provide exposition on monodromy.

Representations of Quivers

Definition 1

A **quiver** is a quadruple $Q = (Q_V, Q_A; s, t)$ where Q_V is called the set of **vertices**, Q_A is called the set of **arrows**, $s : Q_A \to Q_V$ is called the **source map**, $t : Q_A \to Q_V$ is called the **target map**.

Example 2

This is an example of a quiver; note we allow multiple edges and loops.

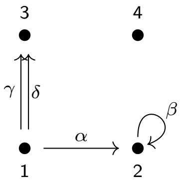

Figure: An example of a quiver Q.

Definition 3

A **quiver representation** $\{(X_i, f_\alpha)\}$ of a quiver Q is an assignment of a vector space X_i to each vertex i and linear map $f_\alpha : X_i \to X_j$ to each arrow $\alpha : i \to j$. The space of representations is $\mathcal{R}\text{ep}(Q, \mathbf{n})$, where $\mathbf{n} = (\dim X_1, ..., \dim X_{|Q_V|})$ is the so-called **dimension vector**.

Remark 4

Choosing bases, we identify the space of representations $\mathcal{R}\mathrm{ep}(Q, \mathbf{n})$ with

$$\prod_{\alpha: i \to j} \mathbb{M}_{\dim(X_j) \times \dim(X_i)}(\mathbb{C}).$$

Definition 5

The **path algebra** of Q is the algebra $\mathbb{C}Q$ generated by arrows α and trivial paths e_i at each vertex i, subject to the following relations:

(i) $e_i^2 = e_i$.

(ii) $e_i e_j = 0$ for $i \neq j$.

(iii) $e_{t(\alpha)}\alpha = \alpha$.

(iv) $\alpha e_{s(\alpha)} = \alpha$.

Remark 6

A **path** in a quiver Q is a concatenation of arrows $p = \alpha_n \cdots \alpha_1$ with $t(\alpha_k) = s(\alpha_{k+1})$. The **length** of the path $\ell(p)$ is the number of arrows n. We read the path right-to-left, so that it is comparable to function composition.

Example 7

The path algebra of the quiver Q with one vertex 1 and one loop α at that vertex is generated by $e_1, \alpha, \alpha^2, \alpha^3, \ldots$. As such, the path algebra $\mathbb{C}Q \cong \mathbb{C}[\alpha]$ is isomorphic to the polynomials in one variable.

Representations of Quivers

Remark 8

Recall that a (left) module M over a ring R is a generalisation of a vector space over a field; it is an Abelian group $(M, +)$ such that there is a map $\cdot : R \times M \to M$ called **scalar multiplication** which is distributive and associative, where the multiplicative identity 1_R acts trivially on M.

Proposition 9

The space $\mathcal{R}ep(Q, \mathbf{n})$ *is equivalent to the space of (left)* $\mathbb{C}Q$*-modules.*

Definition 10

Let Q be a quiver. The **double of** Q is the quiver $\overline{Q}$ obtained by adjoining the reverse arrows $\alpha^* : j \to i$ for each arrow $\alpha : i \to j$.

Example 11

Here, we have a quiver Q which doubles to the quiver $\overline{Q}$.

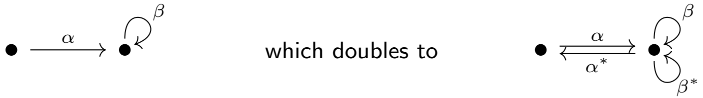

Figure: An example of a quiver with its double.

Double Affine Hecke Algebras

Definition 12

Let $k_0, k_1, u_0, u_1, q \in \mathbb{C}^*$. The **double affine Hecke algebra of type** $C^\vee C_1$ is the algebra $\mathcal{H}_{\mathbf{t},q}$ generated by T_0, T_1, $T_0^\vee$, $T_1^\vee$ subject to these relations:

(i) $(T_0 - k_0)(T_0 + k_0^{-1}) = 0$.
(ii) $(T_1 - k_1)(T_1 + k_1^{-1}) = 0$.
(iii) $(T_0^\vee - u_0)(T_0^\vee + u_0^{-1}) = 0$.
(iv) $(T_1^\vee - u_1)(T_1^\vee + u_1^{-1}) = 0$.
(v) $T_1^\vee T_1 T_0 T_0^\vee = q^{-1/2}$.

Definition 13

The **spherical subalgebra** is $\mathcal{B}_{\mathbf{t},q} = e\mathcal{H}_{\mathbf{t},q}e$, where the **symmetriser** e is

$$e = \frac{1 + k_1 T_1}{1 + k_1^2}.$$

We define the following elements of $\mathcal{H}_{\mathbf{t},1}$:

$$X_1 = T_1^\vee T_1 + T_0 T_0^\vee,$$
$$X_2 = T_1 T_0 + T_0^\vee T_1^\vee,$$
$$X_3 = T_1 T_0^\vee + (T_0^\vee)^{-1} T_1^{-1}.$$

Notation 14

We use the notation $\overline{k}_i := k_i - k_i^{-1}$ and $\overline{u}_i := u_i - u_i^{-1}$.

Theorem 15 ([Obl03, Theorem 2.1, Proposition 2.1])

Let $\mathcal{H}_{\mathbf{t},1}$ be the double affine Hecke algebra with $q = 1$ and define the polynomial $R_{\mathbf{t}} = X_1X_2X_3 - X_1^2 - X_2^2 - X_3^2 + r_1X_1 + r_2X_2 + r_3X_3 + r_4 = 0$ where

$$
\begin{aligned}
r_1 &= \overline{k}_1\overline{u}_1 + \overline{u}_0\overline{k}_0,\\
r_2 &= \overline{k}_0\overline{k}_1 + \overline{u}_0\overline{u}_1,\\
r_3 &= \overline{k}_0\overline{u}_1 + \overline{u}_0\overline{k}_1,\\
r_4 &= \overline{k}_0^2 + \overline{k}_1^2 + \overline{u}_0^2 + \overline{u}_1^2 - \overline{k}_0\overline{k}_1\overline{u}_0\overline{u}_1 + 4.
\end{aligned}
$$

(i) *$Z(\mathcal{H}_{\mathbf{t},1})$ is generated by X_1, X_2, X_3 and $Z(\mathcal{H}_{\mathbf{t},1}) \cong \mathbb{C}[X_1, X_2, X_3]/\langle R_{\mathbf{t}}\rangle$.*

(ii) *We have an isomorphism $\varphi : Z(\mathcal{H}_{\mathbf{t},1}) \to \mathcal{B}_{\mathbf{t},1}$ given by $\varphi(z) = ze$.*

Remark 16

Theorem 15 can be generalised to treat the case where $q \neq 1$, see [Ter13]. The punchline is that $\mathcal{B}_{\mathbf{t},q}$ can be given explicitly by generators and relations.

Double Affine Hecke Algebras

Definition 17

Let G be a group with arbitrary subgroup $H \leq G$ and normal subgroup $N \trianglelefteq G$. Consider the conjugation action $H \times N \to N$ where $h \bullet n = hnh^{-1}$. The **semidirect product** $N \rtimes H$ is the group with underlying set $N \times H$ and almost-pointwise operation $(n_1, h_1) \cdot (n_2, h_2) = (n_1(h_1 \bullet n_2), h_1 h_2)$.

Example 18

Consider the algebra $D_q = \mathbb{C}_q[X^{\pm 1}, P^{\pm 1}] \rtimes \mathbb{C}\mathbb{Z}_2$, which we describe below:

- The elements are linear combinations of $X^i P^j s^\varepsilon$ for $\varepsilon = 0, 1$.
- The multiplication is defined by $sX = X^{-1}s$ and $sP = P^{-1}s$.

We invert the Laurent polynomials in X to form $D_q^{\text{loc}} = \mathbb{C}(X)[P^{\pm 1}] \rtimes \mathbb{C}\mathbb{Z}_2$.

Proposition 19 (Basic Representation)

For all $q \in \mathbb{C}^*$, there is an injective algebra map $\iota_q : \mathcal{H}_{\mathbf{t},q} \hookrightarrow D_q^{loc}$ given by

$$T_0 \mapsto k_0 P^{-1} s + \frac{\overline{k}_0 + \overline{u}_0 X}{1 - X^2}(1 - P^{-1}s), \qquad T_1 \mapsto k_1 s + \frac{\overline{k}_1 + \overline{u}_1 X}{1 - X^2}(1 - s),$$

$$T_0^\vee \mapsto q^{-1/2}\iota_q(T_0)^{-1}X, \qquad T_1^\vee \mapsto X^{-1}\iota_q(T_1)^{-1}.$$

Deligne-Simpson Problem

We consider the **multiplicative Deligne-Simpson problem**: for given conjugacy classes $C_1, \ldots, C_k$ in $\mathrm{GL}_n(\mathbb{C})$, determine solutions of the matrix equation

$$A_1 \cdots A_k = I_n, \qquad \text{where each } A_i \in C_i.$$

Example 20 (Our Situation)

Let $k = 4$ and $n = 2$: we want to solve $A_1 A_2 A_3 A_4 = I_2$ (2×2 matrices here). First, we fix $\xi_{ij} \in \mathbb{C}^*$ for $i = 1, 2, 3, 4$ and $j = 1, 2$ such that the products

$$(A_i - \xi_{i1} I_2)(A_i - \xi_{i2} I_2) = 0.$$

It is clear the ξ_{ij} are eigenvalues of the matrices A_i. The conjugacy classes are then determined by the ranks $n_{i1} = \mathrm{rank}(A_i - \xi_{i1} I_2)$. By convention, we set $n_0 = n = 2$, which allows us to realise as a dimension vector of some quiver representation this collection $\mathbf{n} = (n_0, n_{11}, n_{21}, n_{31}, n_{41})$. Indeed, the quiver in our situation is $\widetilde{D}_4$.

Remark 21

In our situation above, the ranks $n_{i1} = 1$ for all i, so the representation of this quiver will consist of $\mathbb{C}^2$ at the central node and one-dimensional subspaces at the other vertices.

Deligne-Simpson Problem

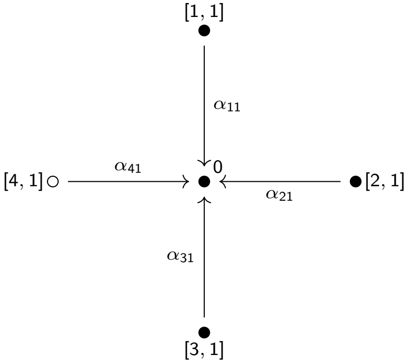

Figure: The affine Dynkin quiver $\widetilde{D}_4$.

Definition 22

Let Q be a quiver. For $\mathbf{q} \in (\mathbb{C}^*)^{|Q_V|}$, the **multiplicative preprojective algebra** is the algebra $\Lambda^{\mathbf{q}}$ satisfying the following properties:

(i) $1 + \alpha\alpha^*$ is invertible in $\Lambda^{\mathbf{q}}$ for all $\alpha \in \overline{Q}_A$.

(ii) $\prod_{\alpha \in \overline{Q}_A} (1 + \alpha\alpha^*)^{\varepsilon(\alpha)} = \sum_{v \in Q_V} q_v e_v$ in $\Lambda^{\mathbf{q}}$, where $\varepsilon(\alpha) = \pm 1$.

Example 23 (Our Situation)

In our situation (from Example 20), let $Q = \widetilde{D}_4$ and define $\mathbf{q} \in (\mathbb{C}^*)^5$ by

$$q_0 := \frac{1}{\xi_{11}\xi_{21}\xi_{31}\xi_{41}}, \qquad q_{i1} := \frac{\xi_{i1}}{\xi_{i2}} \text{ for } i = 1, 2, 3, 4.$$

Then, we have $\Lambda^{\mathbf{q}} \cong \mathbb{C}\overline{Q}/J$, where J is the two-sided ideal generated by these:

$$\begin{aligned}
q_{11}(e_{11} + \alpha_{11}^*\alpha_{11}) &= e_{11},\\
q_{21}(e_{21} + \alpha_{21}^*\alpha_{21}) &= e_{21},\\
q_{31}(e_{31} + \alpha_{31}^*\alpha_{31}) &= e_{31},\\
q_{41}(e_{41} + \alpha_{41}^*\alpha_{41}) &= e_{41},\\
(e_0 + \alpha_{11}\alpha_{11}^*)(e_0 + \alpha_{21}\alpha_{21}^*)(e_0 + \alpha_{31}\alpha_{31}^*)(e_0 + \alpha_{41}\alpha_{41}^*) &= q_0 e_0.
\end{aligned}$$

Deligne-Simpson Problem

Definition 24

Let $\mu, \xi_{i1}, \xi_{i2} \in \mathbb{C}^*$ for $i = 1, 2, 3, 4$. We define the algebra $A_{\mathbf{w},\mu,\boldsymbol{\xi}}$ to be generated by the four elements x_1, x_2, x_3, x_4 subject to these relations:

(i) $(x_1 - \xi_{11}1)(x_1 - \xi_{12}1) = 0$.
(ii) $(x_2 - \xi_{21}1)(x_2 - \xi_{22}1) = 0$.
(iii) $(x_3 - \xi_{31}1)(x_3 - \xi_{32}1) = 0$.
(iv) $(x_4 - \xi_{41}1)(x_4 - \xi_{42}1) = 0$.
(v) $x_1x_2x_3x_4 = \mu$.

Lemma 25

For suitable choice of $(\mathbf{w}, \mu, \boldsymbol{\xi})$, we have an algebra isomorphism $\mathcal{H}_{\mathbf{t},q} \cong A_{\mathbf{w},\mu,\boldsymbol{\xi}}$.

Theorem 26

For suitable multiplicative preprojective algebra $\Lambda^{\mathfrak{q}}$ and idempotent e, we have an algebra isomorphism $A_{\mathbf{w},\mu,\boldsymbol{\xi}} \cong e\Lambda^{\mathfrak{q}}e$.

Proposition 27 ([EOR06, Proposition 11.2])

The algebra $\Lambda^{\mathfrak{q}}/\Lambda^{\mathfrak{q}}e_0\Lambda^{\mathfrak{q}}$ is finite-dimensional. Moreover, if $q_{i1} \neq 1$ for any $i = 1, 2, 3, 4$, then this quotient algebra is zero and thus $\Lambda^{\mathfrak{q}} \overset{\text{Mor}}{\sim} e_0\Lambda^{\mathfrak{q}}e_0$.

Monodromy

Monodromy is the study of an object's behaviour near a singularity (apparently, the word comes from Greek and means 'uniformly running').

Definition 28

The **sixth Painlevé equation** (PVI) is the second-order ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{y(y-1)(y-t)}{t^2(1-t)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right).$$

Remark 29

Full disclosure, Definition 28 frightens me and I am not even close to a differential equations expert, so useful comments are welcome!

Definition 30

Let $A_1, \ldots, A_k$ be $n \times n$ matrices. A **Fuchsian system** is a linear system

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \left(\sum_{i=1}^{k} \frac{A_i}{\lambda - t_i}\right)\Phi,$$

where variable $\lambda \in \mathbb{CP}^1 \setminus \{t_1, \ldots, t_k\}$ and Φ is a $\mathbb{M}_{n\times n}(\mathbb{C})$-valued function of λ.

Monodromy

Remark 31

Recall that $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, the one-point compactification of the complex plane. In Definition 30, if we impose the *additive* Deligne Simpson condition

$$A_1 + \cdots + A_k = 0,$$

then the point at infinity $\lambda = \infty$ is a so-called *regular point* of the system; any neighbourhood of this point contains a solution of the Fuchsian system with the condition that $\Phi|_{\lambda=\infty} = I_n$, the identity matrix.

Example 32 (Our Situation)

Let A_1, A_2, A_3 be 2×2 matrices and consider the punctured Riemann sphere $X := \mathbb{CP}^1 \setminus \{0, 1, t, \infty\}$. The Fuchsian system of interest is the following:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1} + \frac{A_3}{\lambda - t} \right) \Phi.$$

We define the matrix corresponding to infinity as $A_\infty := -A_1 - A_2 - A_3$.

Remark 33

For $i = \infty, 1, 2, 3$, the eigenvalues depend on $\alpha, \beta, \gamma, \delta$ from (PVI) respectively.

Monodromy

We say that the Fuchsian system encodes the monodromy data for the punctured space X. Let Φ be a solution to the Fuchsian system in Example 32 where it is the identity at $\lambda = \infty$ and the matrices sum to zero.

Remark 34

Recall the fundamental group $\pi_1(X, x)$ consists of classes of loops at $x \in X$.

Choosing some $x \in X$ and considering a loop based here that encircles the poles $0, 1, t, \infty$, we get a representation $\rho : \pi_1(X, x) \to \mathrm{GL}_2(\mathbb{C})$. The matrices M_1, M_2, M_3 correspond to the loops around each of the points $0, 1, t$ respectively.

We can choose the orientation so that the generators are path-homotopic to the loop at infinity, meaning we get the *multiplicative* Deligne-Simpson condition

$$M_\infty = (M_1 M_2 M_3)^{-1}.$$

This is the multiplicative analogue of the expression for A_∞; monodromy allows us to pass between the multiplicative and additive Deligne-Simpson problems.

In [CM10], the $M_i \in \mathrm{GL}_2(\mathbb{C})$ are given locally as follows:

$$M_i \sim \exp(A_i) \sim \begin{pmatrix} e^{\theta_i/2} & 0 \\ 0 & -e^{-\theta_i/2} \end{pmatrix}.$$

Representation of the Double Affine Hecke Algebra via Matrices

Let $A_q := \mathbb{C}_q[X^{\pm 1}, P^{\pm 1}]$. We have an A_q-action on $\mathbb{C}[X^{\pm 1}]$ given by

$$X{\bullet}X^j = X^{j+1}, \qquad P{\bullet}X^j = q^j X.$$

This gives a representation $\pi : A_q \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X^{\pm 1}])$. Alright, but we want to get a representation of D_q from earlier; this is achieved by the induced representation of $D_q = A_q \rtimes \mathbb{C}\mathbb{Z}_2$ via π, which is defined as follows:

$$\mathrm{Ind}_{A_q}^{D_q}(\pi) := D_q \otimes_{A_q} \pi \cong \mathbb{C}\mathbb{Z}_2 \otimes \mathbb{C}[X^{\pm 1}] \cong \mathbb{C}[X^{\pm 1}]^{\oplus 2}.$$

Thus, we get a representation $\hat{\pi} : D_q \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X^{\pm 1}]^{\oplus 2})$ defined by these:

$$\begin{aligned} X \bullet (f_1 + sf_2) &= X{\bullet}f_1 + sX^{-1}{\bullet}f_2, \\ P \bullet (f_1 + sf_2) &= P{\bullet}f_1 + sP^{-1}{\bullet}f_2, \\ s \bullet (f_1 + sf_2) &= f_2 + sf_1. \end{aligned}$$

Remark 35

In the same way, we extend this representation to all of $\mathbb{C}(X)$, that is we get a representation $\hat{\pi} : D_q^{\mathrm{loc}} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}(X)^{\oplus 2})$, where the acting formulae and the subsequent matrix formulae are preserved in this localised situation.

Notation 36

We abuse Basic Representation notation and let $T_i := \iota_q(T_i)$, $T_i^{\vee} := \iota_q(T_i^{\vee})$.

Representation of the Double Affine Hecke Algebra via Matrices

We can define the representation $\hat{\pi}$ by matrices in the basis $\{1, s\}$:

$$\hat{\pi}(X) = \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix}, \qquad \hat{\pi}(P) = \begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix}, \qquad \hat{\pi}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note also that for any rational function $f \in \mathbb{C}(X)$, we have

$$\hat{\pi}(f(X)) = \begin{pmatrix} f(X) & 0 \\ 0 & f(X^{-1}) \end{pmatrix}.$$

We are now in a position to exploit the Basic Representation to generate some 2×2 matrices which represent the type $C^{\vee}C_1$ double affine Hecke algebra. Indeed, we need only decompose the T_i into the forms $T_0 = a_0(X) + b_0(X)P^{-1}s$ and $T_1 = a_1(X) + b_1(X)s$, as is done below:

$$T_0 = \frac{\overline{k}_0 + \overline{u}_0 X}{1 - X^2} + \left(k_0 - \frac{\overline{k}_0 + \overline{u}_0 X}{1 - X^2} \right) P^{-1} s,$$

$$T_1 = \frac{\overline{k}_1 + \overline{u}_1 X}{1 - X^2} + \left(k_1 - \frac{\overline{k}_1 + \overline{u}_1 X}{1 - X^2} \right) s.$$

Representation of the Double Affine Hecke Algebra via Matrices

Hence, we find the corresponding matrices:

$$\hat{\pi}(T_0) = \begin{pmatrix} \dfrac{\overline{k}_0 + \overline{u}_0 X}{1 - X^2} & k_0 P^{-1} - \dfrac{\overline{k}_0 + \overline{u}_0 X}{1 - X^2} P^{-1} \\ k_0 P + \dfrac{\overline{k}_0 X^2 + \overline{u}_0 X}{1 - X^2} P & -\dfrac{\overline{k}_0 X^2 + \overline{u}_0 X}{1 - X^2} \end{pmatrix},$$

$$\hat{\pi}(T_1) = \begin{pmatrix} \dfrac{\overline{k}_1 + \overline{u}_1 X}{1 - X^2} & k_1 - \dfrac{\overline{k}_1 + \overline{u}_1 X}{1 - X^2} \\ k_1 + \dfrac{\overline{k}_1 X^2 + \overline{u}_1 X}{1 - X^2} & -\dfrac{\overline{k}_1 X^2 + \overline{u}_1 X}{1 - X^2} \end{pmatrix}.$$

We easily represent the $T_i^\vee$ by matrices using the expressions we have for $\hat{\pi}(T_i)$ in conjunction with the Basic Representation. First, note that

$$\hat{\pi}(X)^{-1} = \begin{pmatrix} X^{-1} & 0 \\ 0 & X \end{pmatrix}.$$

Representation of the Double Affine Hecke Algebra via Matrices

We now have all we require to compute the remaining matrices:

$$\hat{\pi}(T_0^\vee) = q^{-1/2}\begin{pmatrix} -\dfrac{\overline{k}_0 X + \overline{u}_0 X^2}{1-X^2} & -k_0 P^{-1}X^{-1} + \dfrac{\overline{k}_0 + \overline{u}_0 X}{1-X^2}P^{-1}X^{-1} \\ -k_0 PX - \dfrac{\overline{k}_0 X^2 + \overline{u}_0 X}{1-X^2}PX & \dfrac{\overline{k}_0 X + \overline{u}_0}{1-X^2} \end{pmatrix},$$

$$\hat{\pi}(T_1^\vee) = \begin{pmatrix} -\dfrac{\overline{k}_1 X^{-1} + \overline{u}_1}{1-X^2} & -k_1 X^{-1} + \dfrac{\overline{k}_1 X^{-1} + \overline{u}_1}{1-X^2} \\ -k_1 X - \dfrac{\overline{k}_1 X^3 + \overline{u}_1 X^2}{1-X^2} & \dfrac{\overline{k}_1 X^3 + \overline{u}_1 X^2}{1-X^2} \end{pmatrix}.$$

Notation 37

We label the matrices $\hat{\pi}(T_1^\vee) =: A_1, \hat{\pi}(T_1) =: A_2, \hat{\pi}(T_0) =: A_3, \hat{\pi}(T_0^\vee) =: A_4$.

The representation $\hat{\pi}$ respects the relations of the double affine Hecke algebra; this means that $A_1A_2A_3A_4 = q^{-1/2}I_2$ and $(A_i - t_iI_2)(A_i + t_i^{-1}I_2) = 0$, for the relevant parameters t_i in Definition 12.

Conclusions

- For $q = 1$, the four quadratic relations remain but the product relation reduces to $A_1 A_2 A_3 A_4 = I_2$, the multiplicative Deligne-Simpson problem.
- Looking at the matrices $\hat{\pi}(T_i)$ and $\hat{\pi}(T_i^\vee)$, we have a two-parameter family of solutions to the Deligne-Simpson problem, parametrised by X and P.
- In quiver terms, this provides the following representation of $\widetilde{D}_4$: the one-dimensional spaces $\mathrm{im}(A_i - t_i I_2)$ are on the outside vertices, $\mathbb{C}^2$ is at the central node and the linear maps are inclusions.
- From another perspective, we can associate to each A_i a monodromy matrix; there is ambiguity in that M_∞ is determined by the other M_i, so one must choose which of our matrices A_i corresponds to this M_∞. In this way, we see that the eigenvalues $t_i = e^{\theta_i/2}$.
- The matrices can be used to parametrise the affine cubic surface $R_{\mathbf{t}} = 0$.

Problem 38

The matrices have poles at $X^2 = 1$. Is it possible to degenerate, as $X \to 1$, to get solutions to the Deligne-Simpson problem for $X = 1$? If not, we may not have a global system of coordinates on the affine cubic surface; points of the form $(1, P)$ may be missed in this (X, P)-parametrisation.

Future Work

- Study the double affine Hecke algebra of type $C^\vee C_n$, that is an algebra defined similarly to Definition 12 except with generators $T_0, \dots, T_n$ subject to braid relations, the quadratic relations, and more.
- Derive a similar story for the Basic Representation in the type $C^\vee C_n$ situation following work by [EGO06] and [Cha19] and convert this theory into matrices solving an analogous Deligne-Simpson problem.
- Use this to get a parametrisation of an affine cubic surface in the type $C^\vee C_n$ situation and describe the spherical subalgebra à la Theorem 15.

Thanks for Listening!

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