Knots and the Fundamental Group

Bradley Ryan

Pure Mathematics Postgraduate Research Seminar, University of Leeds

06/05/2021

Introduction to Knots

Definition

A **topology** on a set X is a set τ of subsets of X satisfying the following:

T1. $\emptyset, X \in \tau$.

T2. τ is closed under finite intersections.

T3. τ is closed under arbitrary unions.

Members of τ are **open** in X. A subset is **closed** in X if its complement is open.

Example

The set $\mathbb{R}^n$ is a topological space when equipped with the standard Euclidean topology, that is the one generated by open balls, that is the open sets in $\mathbb{R}^n$ are arbitrary unions of $B_\varepsilon(x) := \{y \in \mathbb{R}^n : |x - y| < \varepsilon\}$ for every $\varepsilon > 0$ and $x \in \mathbb{R}^n$.

Remark

Open and closed are **not** opposites; a set can be one of them, both or neither.

Definition

Let X and Y be topological spaces. A map $f : X \to Y$ is **continuous** if, for any $U \subseteq Y$ open, the pre-image $f^{-1}(U) \subseteq X$ is open. It is a **homeomorphism** if it is a continuous bijection with continuous inverse. An **embedding** is a continuous injection which is a homeomorphism onto its image $f(X)$.

Introduction to Knots

Definition

Let $f, g : X \to Y$ be embeddings of topological spaces. An **ambient isotopy** from f to g is a continuous map $F : Y \times [0,1] \to Y$ that is a homeomorphism for every $t \in [0,1]$ and satisfies the following, for all $y \in Y$ and $x \in X$:

$$F(y,0) = y \qquad \text{and} \qquad F(f(x),1) = g(x).$$

Lemma

Ambient isotopy $\simeq$ is an equivalence relation.

Definition

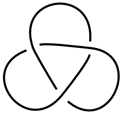
A closed parametrised curve is called **knotted** if it is not ambient isotopic to the circle S^1. Otherwise, it is called **unknotted**.

A **knot** $K \subseteq \mathbb{R}^3$ is a class of closed curves under the equivalence relation ambient isotopy. One of the most effective ways to picture a knot comes about by using **knot diagrams**; these are projections of a knot K onto the plane $\mathbb{R}^2$ such that each crossing consists of a continuous curve (over-crossing) and a discontinuous curve (under-crossing).

Introduction to Knots

Example

The simplest non-trivial knot is the trefoil and its projection is given below.



Definition

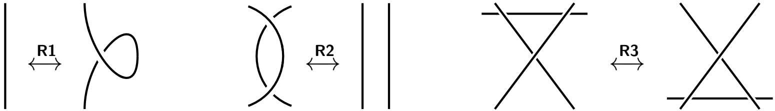
The **Reidemeister moves** are the following, applied locally to a knot's arcs:

R0. Planar isotopy.

R1. Twisting part of an arc.

R2. Moving an arc in front of or behind another arc.

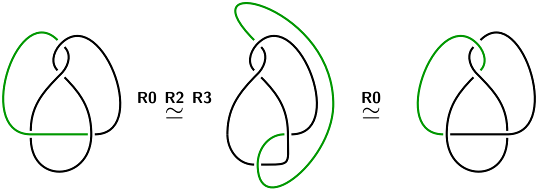
R3. Moving an arc in front of or behind a crossing.



Introduction to Knots

Example

Applying Reidemeister moves to transform the green arc of the figure-eight knot so that it passes between chirality.



This example shows that the figure-eight knot is isotopic to its mirror image. This is not true for all knots (the trefoil is not; proving so requires more machinery).

Definition

A knot K is **achiral** if it is isotopic to its mirror image $\overline{K}$. Otherwise, it is **chiral**.

How can we prove results on chirality and other knot invariants without explicitly parametrising a closed curve and computing an ambient isotopy to see (i) whether or not the curve is knotted, and (ii) if it is ambient isotopic to another knot?

Introduction to Knots

We breathe a sigh of relief and state one of knot theory's most important results.

Theorem (Reidemeister's Theorem)

Two knots K_1 and K_2 are ambient isotopic if and only if their diagrams are related by a finite sequence of Reidemeister moves.

Remark

The nomenclature here can make Reidemeister's Theorem seem trivial but it really isn't; we say knot diagrams are *isotopic* when we can pass between them using Reidemeister moves whereas *ambient isotopic* is the topological notion defined at the beginning.

Example

We now know that the figure-eight knot is **ambient** isotopic to its mirror image, that is they lie in the same ambient isotopy class and so truly do define the same knot.

Knot Colourings

One simple way to prove that the trefoil knot is indeed a knotted curve is to consider p-colourability, a colouring of the arcs of the trefoil with p (odd prime) distinct colours under certain rules.

Definition

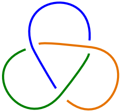
Let $p > 2$ be prime. A knot diagram is ***p*-colourable** if it can be coloured with p colours such that the following are satisfied:

(i) At least two colours are used.

(ii) For $x, y, z \in \mathbb{Z}_p$ colours at a crossing with over-crossing z and under-crossings x and y, it must be that $x + y - 2z \equiv 0 \pmod{p}$.

Example

An example of a three-colouring of one diagram of the trefoil.



Theorem

p-colourability is an isotopy invariant.

Knot Colourings

Definition

Let K be a diagram of a knot with n crossings. A **colouring matrix** M_K of K is an $n \times n$ matrix with rows and columns representing the crossing and arcs of K, respectively, where an entry of 2 denotes an over-crossing, an entry of -1 denotes an under-crossing and an entry of 0 denotes exclusion from a crossing.

Example

The colouring matrix of the trefoil knot K, with respect to a given labelling, is

$$M_K = \begin{pmatrix} -1 & -1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}.$$

Definition

Let K be a diagram of a knot. The **knot determinant** $\det(K)$ of K is the absolute value of *any* minor of the matrix M_K. By convention, $\det(S^1) = 1$.

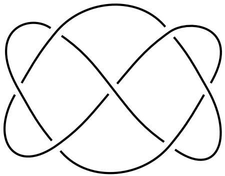
Proposition

Let K be a diagram of a knot. K is p-colourable if and only if $p \mid \det(K)$.

Knot Colourings

Example

Consider the below diagram of the endless knot K and its colouring matrix:



$$\begin{pmatrix} -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \\ 2 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 2 & 0 & 0 & -1 \end{pmatrix}$$

One can compute the knot determinant to be $\det(K) = 15$. The proposition above guarantees that it is both 3-colourable and 5-colourable.

Exercise

If the talk is knot interesting, pass the time by finding both a 3-colouring and a 5-colouring of the endless knot K (if you are **very** bored, verify $\det(K) = 15$).

The Knot Group

We now detour into algebraic topology which will not only give us a powerful knot invariant but will allow me to live up to this seminar's title!

Definition

Let X be a topological space. A **path** is a continuous map $\alpha : [0,1] \to X$. A path is a **loop based at** p if $\alpha(0) = p = \alpha(1)$.

Definition

Let $\alpha, \beta : [0,1] \to X$ be paths with endpoints x and y. We call α and β **path-homotopic** if there is a continuous $F : [0,1] \times [0,1] \to X$ such that the following conditions hold:

(i) $F(s,0) = \alpha(s)$.

(ii) $F(s,1) = \beta(s)$.

(iii) $F(0,t) = x$.

(iv) $F(1,t) = y$.

Remark

We can also join two paths together given that the start-point of the second path is the endpoint of the first. Furthermore, we can trace a path in the reverse direction and we can even define the trivial path at a single point. This gives us a group-like structure.

The Knot Group

Definition

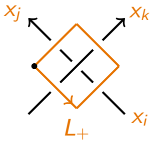
Let X be a topological space and $x \in X$. The **fundamental group** is the set $\pi_1(X, x) = \{[\alpha] : \alpha \text{ is a loop based at } x\}$ under the join operation.

Definition

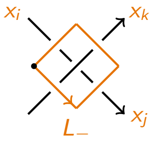
Let K be a knot. The **knot group** is the fundamental group $\pi_1(\mathbb{R}^3 \setminus K)$.

Remark

An alternate approach is to define the so-called **Wirtinger presentation** of the knot group, which has generators x_i corresponding to arcs of K and relations based on the interaction of positive crossings L_+ and negative crossings L_-:



$x_k x_i x_k^{-1} x_j^{-1} = 1$



$x_k x_j^{-1} x_k^{-1} x_i = 1$

The Knot Group

Theorem

The knot group is an isotopy invariant.

Proof

Homeomorphic spaces have isomorphic fundamental groups. As such, for $K_1 \simeq K_2$, it follows $\mathbb{R}^3 \setminus K_1 \cong \mathbb{R}^3 \setminus K_2$ and so $\pi_1(\mathbb{R}^3 \setminus K_1) \cong \pi_1(\mathbb{R}^3 \setminus K_2)$. □

Proposition

Let K be a diagram of a knot. Then, K is p-colourable if and only if there exists a group homomorphism $\varphi : \pi_1(\mathbb{R}^3 \setminus K) \to D_{2p}$ to the dihedral group.

Proof

The dihedral group $D_{2p} = \langle r, s; r^p, s^2, rsrs \rangle$. Let $\varphi(x_i) = r^{c_i} s$, where c_i is the colour of the arc x_i; apply this to the (first) Wirtinger relation:

$$\begin{aligned} & \varphi(1) = \varphi(x_k x_i x_k^{-1} x_j^{-1}) \\ \Leftrightarrow \quad & 1 = r^{c_k} s r^{c_i} s s^{-1} r^{-c_k} s^{-1} r^{-c_j} \\ & = r^{c_k} r^{-c_i} r^{c_k} r^{-c_j} \\ & = r^{2c_k - c_i - c_j} \\ \Leftrightarrow \quad & c_i + c_j - 2c_k \equiv 0 \pmod{p}. \end{aligned}$$

□

The ‘End’

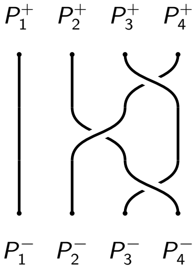
Brief Braid Theory

Definition

A **braid on n-strings** consists of n arcs $x_1, ..., x_n$ with integer endpoints $1, ..., n$. More specifically, each x_i connects a point $P_i^+ = (i, 0, 1)$ in the so-called upper plane to a point $P_{\pi(i)}^- = (i, 0, 0)$ in the so-called lower plane, where $\pi \in S_n$ is a permutation. In the case that π is trivial, we call it a **pure braid on n-strings**.

Example

Consider the following braid diagram, in which the permutation $\pi = (1)(3)(24)$.



Convention: if an arc goes **over** from P_i to P_{i+1}, then it is represented by σ_i but if an arc goes **over** from P_{i+1} to P_i, then it is represented by σ_i^{-1}.

Brief Braid Theory

Definition

The **Artin moves** are the following, applied to the arcs of a braid on n-strings:

A1. $\sigma_i \sigma_j = \sigma_j \sigma_i$ whenever $|i - j| > 1$.

A2. $\sigma_i \sigma_i^{-1} = 1 = \sigma_i^{-1} \sigma_i$.

A3. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

Remark

- This gives rise to the **Artin presentation of the braid group**. For a braid on n-strands, the associated braid group is that with $n - 1$ generators subject to relations given by the Artin moves. Call this group B_n.
- The symmetric group S_n has the so-called **Coxeter presentation** on generators $s_i = (i, i + 1)$ transpositions with relations given by the Artin moves. The pure braid group on n-strands is then the kernel of the group homomorphism $B_n \to S_n$ where $\sigma_i \mapsto s_i$.
- The braid group can be given as the fundamental group of a certain space.

Theorem (Alexander's Theorem)

Every knot is isotopic to a closed (identify the upper and lower planes) braid.

A Different Direction

In rather naïve terms, a **complex Coxeter group** is a group which is generated by 'reflections' (e.g. $W = \mathbb{Z}_2$) and the **Cherednik algebra** $H_{t,c}$ of a Coxeter group is the algebra generated by the algebras $\mathbb{C}W$, $\mathbb{C}[\mathfrak{h}]$, $\mathbb{C}[\mathfrak{h}^*]$, where $\mathfrak{h}$ is a complex vector space (e.g. $\mathfrak{h} = \mathbb{C}^n$), subject to some relations.

My research is heading down the following avenue: there is a generalisation called the **double affine Hecke algebra** $\mathcal{H}_{\mathbf{t},q}$; I hope we can reconvene down the line to continue this story in the more algebraic setting (when I can better convince you that I know what I'm talking about).

Although this is a far-cry from the title of the talk, it is interesting to see knot theory have an impact on some very algebraic structures!

The True End