# Double Affine Hecke Algebras and Calogero-Moser Spaces

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# Hecke Algebras of Type $C_n$

## Definition 1

Let $\mathbf{t} = (k_0, k_n, t) \in (\mathbb{C}^*)^3$. The **affine Hecke algebra** of type $\widetilde{C}_n$ is the algebra $\widetilde{H}_{n,\mathbf{t}}$ generated by $T_0, T_1, ..., T_n$ satisfying the following:

$$
\begin{aligned}
[T_i, T_j] &= 0, && |i-j| > 1 \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, && \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && i = 1, ..., n-2 \\
T_{n-1} T_n T_{n-1} T_n &= T_n T_{n-1} T_n T_{n-1}, && \\
(T_0 - k_0)(T_0 + k_0^{-1}) &= 0, && \\
(T_i - t)(T_i + t^{-1}) &= 0, && i = 1, ..., n-1 \\
(T_n - k_n)(T_n + k_n^{-1}) &= 0. &&
\end{aligned}
$$

## Remark 2

The **Hecke algebra** of type $C_n$ is the subalgebra $H_{n,\mathbf{t}}$ generated by $T_1, ..., T_n$.

# Generators of the Affine Hecke Algebra

**Approach 1:** The affine Hecke algebra is generated by $T_0, T_1, ..., T_n$.

**Approach 2:** The affine Hecke algebra is generated by $T_1, ..., T_n$ and $Y_1, ..., Y_n$.

Remark 3

We have an explicit formula $Y_i := T_i \cdots T_{n-1} T_n T_{n-1} \cdots T_1 T_0 T_1^{-1} \cdots T_{i-1}^{-1}$.

Proposition 4 ([Lusztig])

*The $Y_i$ pairwise commute and generate a subalgebra $\mathbb{C}[\boldsymbol{Y}^{\pm 1}]$ such that*

$$\widetilde{H} \cong H \otimes \mathbb{C}[\boldsymbol{Y}^{\pm 1}].$$

## The Double Affine Hecke Algebra of Type $C^\vee C_n$

### Definition 5 ([Sahi])

Let $q \in \mathbb{C}^*$ and $\mathbf{t} = (k_0, k_n, t, u_0, u_n) \in (\mathbb{C}^*)^5$. The **DAHA of type** $C^\vee C_n$ is the algebra $\mathcal{H}_{n,\mathbf{t},q}$ generated by $T_0, T_1, ..., T_n$ and $X_1, ..., X_n$ satisfying the following:

$$
\begin{aligned}
[T_i, T_j] &= 0, && |i-j| > 1\\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, &&\\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && i = 1, ..., n-2\\
T_{n-1} T_n T_{n-1} T_n &= T_n T_{n-1} T_n T_{n-1}, &&\\
[X_i, X_j] &= 0, && 1 \leq i,j \leq n\\
[T_i, X_j] &= 0, && j \neq i, i+1\\
T_i X_i T_i &= X_{i+1}, && i = 1, ..., n-1\\
(T_0 - k_0)(T_0 + k_0^{-1}) &= 0, &&\\
(T_i - t)(T_i + t^{-1}) &= 0, && i = 1, ..., n-1\\
(T_n - k_n)(T_n + k_n^{-1}) &= 0, &&\\
(T_0^\vee - u_0)(T_0^\vee + u_0^{-1}) &= 0, && T_0^\vee := q^{-1} T_0^{-1} X_1\\
(T_n^\vee - u_n)(T_n^\vee + u_n^{-1}) &= 0. && T_n^\vee := X_n^{-1} T_n^{-1}
\end{aligned}
$$

# Spherical Subalgebra

### Definition 6

The **spherical subalgebra** of the DAHA is the subalgebra $\mathbf{e}\mathcal{H}_{n,\mathbf{t},q}\mathbf{e}$, where

$$\mathbf{e} := \frac{1}{\sum_{w \in W} \tau_w^2} \sum_{w \in W} \tau_w T_w.$$

### Theorem 7

*When $q = 1$, we have an isomorphism $Z(\mathcal{H}_{n,\mathbf{t},1}) \cong \mathbf{e}\mathcal{H}_{n,\mathbf{t},1}\mathbf{e}$ given by $z \mapsto z\mathbf{e}$.*

### Problem 8

*Find a variety $V$ such that the centre $Z$ of the DAHA $\mathcal{H} := \mathcal{H}_{n,\mathbf{t},1}$ when $q = 1$ is isomorphic to the algebra of functions on said variety, that is $Z \cong \mathbb{C}[V]$.*

### Definition 9

Fix conjugacy classes $\mathcal{C}_1, ..., \mathcal{C}_k \subseteq \mathrm{GL}_m(\mathbb{C})$, an integer $g \geq 0$ and define the set

$$\widehat{\mathcal{M}} := \{A_1, B_1, ..., A_g, B_g \in \mathrm{GL}_m(\mathbb{C}), C_i \in \mathcal{C}_i : (A_1 B_1 A_1^{-1} B_1^{-1}) \cdots (A_g B_g A_g^{-1} B_g^{-1}) C_1 \cdots C_k = \mathbb{1}_m\}.$$

The corresponding $\mathrm{GL}_m$-**character variety** is defined as follows:

$$\mathcal{M}_{g,k} := \widehat{\mathcal{M}} /\!\!/ \mathrm{PGL}_m(\mathbb{C}).$$

### Definition 10

Fix semi-simple conjugacy classes in $\mathrm{GL}_m(\mathbb{C})$ by specifying eigenvalues $\boldsymbol{\lambda}$ and multiplicities $\boldsymbol{\mu}$. This data is said to be **generic** if for any $1 \leq s < m$ and sub-multiplicities $\nu_{ij} \leq \mu_{ij}$ where $\nu_{i1} + \cdots + \nu_{i\ell_i} = s$ for each $i$, we have

$$\prod_{i=1}^{k} \prod_{j=1}^{\ell_i} \lambda_{ij}^{\mu_{ij}} = 1 \qquad \text{and} \qquad \prod_{i=1}^{k} \prod_{j=1}^{\ell_i} \lambda_{ij}^{\nu_{ij}} \neq 1.$$

# Generic Semi-Simple Eigendata

For generic semi-simple conjugacy classes, we have these (if non-empty):

- $\mathcal{M}_{g,k}$ is smooth.
- $\mathcal{M}_{g,k}$ is $d_{\boldsymbol{\mu}}$-equidimensional where $d_{\boldsymbol{\mu}} := (2g-2+k)m^2 - \sum_{i=1}^{k}\sum_{j=1}^{\ell_i}\mu_{ij}^2 + 2.$

Theorem 11 ([Hausel, Letellier and Rodriguez-Villegas])

*For generic semi-simple conjugacy classes, the variety $\mathcal{M}_{g,k}$ is connected.*

# Calogero-Moser Space in Type $C^\vee C_n$

Fix the following generic semi-simple conjugacy classes in $\mathrm{GL}_{2n}(\mathbb{C})$:

$$\mathcal{C}_1 = [\mathrm{diag}(\underbrace{k_0, \dots, k_0}_{n}, \underbrace{-k_0^{-1}, .., -k_0^{-1}}_{n})],$$

$$\mathcal{C}_2 = [\mathrm{diag}(\underbrace{u_0, \dots, u_0}_{n}, \underbrace{-u_0^{-1}, .., -u_0^{-1}}_{n})],$$

$$\mathcal{C}_3 = [\mathrm{diag}(\underbrace{u_n, \dots, u_n}_{n}, \underbrace{-u_n^{-1}, .., -u_n^{-1}}_{n})],$$

$$\mathcal{C}_4 = [\mathrm{diag}(\underbrace{-k_n^{-1}, \dots, -k_n^{-1}}_{n}, \underbrace{k_n t^{-2}, \dots, k_n t^{-2}}_{n-1}, \underbrace{k_n t^{2n-2}}_{1})].$$

## Definition 12

The **Calogero-Moser space** is character variety of the four-punctured sphere:

$$\mathrm{CM} := \{(A_1, A_2, A_3, A_4) : A_i \in \mathcal{C}_i \text{ and } A_1 A_2 A_3 A_4 = \mathbb{1}_{2n}\} / \mathrm{GL}_{2n}(\mathbb{C}).$$

Theorem (conjectured in [Etingof, Gan and Oblomkov])

*The centre Z of the DAHA* $\mathcal{H} := \mathcal{H}_{n,\mathbf{t},1}$ *when* $q = 1$ *is isomorphic to the algebra of functions on the Calogero-Moser space, that is* $Z \cong \mathbb{C}[\mathrm{CM}]$.

# A Map from the DAHA to Calogero-Moser Space

Proposition 13 (cf. [Etingof, Gan and Oblomkov])

*There is an explicit map* $\Phi : \mathrm{Spec}(Z) \to \mathrm{CM}$.

The correspondence between matrices and DAHA elements via $\Phi$ is as follows:

$$A_1 \leftrightarrow T_0, \qquad A_2 \leftrightarrow T_0^\vee, \qquad A_3 \leftrightarrow S T_n^\vee S^{-1}, \qquad A_4 \leftrightarrow S T_n S^\dagger.$$

Problem 14

*An explicit inverse is completely unknown; this is an open and difficult problem.*

## Multiplicative Quiver Varieties

### Definition 15 ([Crawley-Boevey and Shaw])

Fix $\mathbf{q} = (q_v) \in (\mathbb{C}^*)^{Q_0}$. The **multiplicative preprojective algebra** is

$$\Lambda^{\mathbf{q}} \coloneqq \mathbb{C}\overline{Q}[(1 + a^* a)^{-1}] \Big/ \left\langle \prod_{a \in \overline{Q}_1} (1 + a^* a)^{\varepsilon(a)} - \sum_{v \in Q_0} q_v e_v \right\rangle .$$

### Remark 16

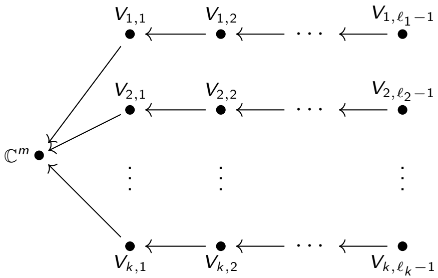
Identify $\operatorname{Rep}(\Lambda^{\mathbf{q}}, \boldsymbol{\alpha})$ with a subset of $\operatorname{Rep}(\overline{Q}, \boldsymbol{\alpha})$ where the relations are akin to those in $\Lambda^{\mathbf{q}}$ except the role of the arrows is now replaced by linear maps.

### Definition 17

The **multiplicative quiver variety** associated with $Q$ is $\operatorname{Rep}(\Lambda^{\mathbf{q}}, \boldsymbol{\alpha}) /\!\!/ G(\boldsymbol{\alpha})$, where $G(\boldsymbol{\alpha}) \coloneqq (\prod_v \mathrm{GL}_{\alpha_v})/\mathbb{C}^*$ acts on a representation by conjugation.

# From Character to Quiver Varieties

Let $\mathcal{M}_{0,k}$ be a character variety with semi-simple generic conjugacy classes. One can associate to it a quiver variety with this underlying *star-shaped* quiver:

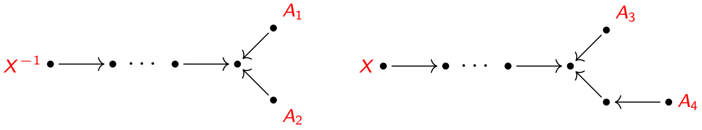



- The vector spaces are $V_{i,j} := \mathrm{im}\left((A_i - \lambda_{i1}) \cdots (A_i - \lambda_{ij})\right)$.

- The numbers are $q_0 := \prod_{i=1}^{k} \frac{1}{\lambda_{i1}}$ and $q_{[i,j]} := \frac{\lambda_{ij}}{\lambda_{i\,j+1}}$.

# Local Coordinates on Calogero-Moser Space

Solving $A_1 A_2 A_3 A_4 = \mathbb{1}_{2n}$ where $A_i \in \mathcal{C}_i$ can be split into two related problems.




(a) The quiver $Q_1$ for $A_1 A_2 X^{-1} = \mathbb{1}_{2n}$.

(b) The quiver $Q_2$ for $X A_3 A_4 = \mathbb{1}_{2n}$.

**Theorem 18**

*The map* $\Phi$ *restricts to an isomorphism* $\mathrm{Spec}(Z_{\delta(\boldsymbol{x})}) \xrightarrow{\sim} \mathrm{CM}_{\delta(\boldsymbol{x})}$.

# The Matrix for $A_1$

This is the matrix $A_1$ on the Calogero-Moser subset $\mathrm{CM}_{\delta(\boldsymbol{X})}$ in local coordinates:

$$A_1 = \left(\begin{array}{ccc|ccc} a_1^- & & & b_1^- & & \\ & \ddots & & & \ddots & \\ & & a_n^- & & & b_n^- \\ \hline b_1 & & & a_1 & & \\ & \ddots & & & \ddots & \\ & & b_n & & & a_n \end{array}\right),$$

$$a_i = \frac{(k_0 - k_0^{-1}) + (u_0 - u_0^{-1})X_i}{1 - X_i^2},$$

$$b_i = k_0 P_i - \frac{(k_0 - k_0^{-1}) + (u_0 - u_0^{-1})X_i}{1 - X_i^2} P_i,$$

$$a_i^- := a_i(X_i^{-1}),$$

$$b_i^- := b_i(P_i^{-1}, X_i^{-1}).$$

# The Matrix for $A_2$

This is the matrix $A_2$ on the Calogero-Moser subset $\mathrm{CM}_{\delta(\boldsymbol{X})}$ in local coordinates:

$$A_2 = \left(\begin{array}{ccc|ccc} c_1 & & & d_1^- & & \\ & \ddots & & & \ddots & \\ & & c_n & & & d_n^- \\ \hline d_1 & & & c_1^- & & \\ & \ddots & & & \ddots & \\ & & d_n & & & c_n^- \end{array}\right),$$

$$c_i = -\frac{(k_0 - k_0^{-1}) + (u_0 - u_0^{-1})X_i}{1 - X_i^2} X_i,$$

$$d_i = k_0 P_i X_i - \frac{(k_0 - k_0^{-1}) + (u_0 - u_0^{-1})X_i}{1 - X_i^2} P_i X_i,$$

$$c_i^- := c_i(X_i^{-1}),$$

$$d_i^- := d_i(P_i^{-1}, X_i^{-1}).$$

# The Matrix for $A_3$

This is the matrix $A_3$ on the Calogero-Moser subset $\mathrm{CM}_{\delta(\boldsymbol{X})}$ in local coordinates:

$$(A_3)_{ij} = \begin{cases} e_j^- \prod_{\substack{k=1 \\ k-j\neq 0,\pm n}}^{2n} a_{jk}^-, & \text{if } i-j=\pm n \\ e_j^- b_{ij}^+ a_{ij} \prod_{\substack{k=1 \\ k-j\neq 0,\pm n}}^{2n} a_{jk}^-, & \text{if } i-j\neq 0,\pm n \\ k_n^{-1}X_i^{-1} - \sum_{k\neq j}(A_3)_{ik}, & \text{if } i=j \end{cases},$$

$$a_{ij} = \frac{t^{-1} - tX_iX_j^{-1}}{1 - X_iX_j^{-1}},$$

$$b_{ij} = \frac{t - t^{-1}}{1 - X_iX_j^{-1}},$$

$$e_i = \frac{k_n^{-1} - k_nX_i^2 - (u_n - u_n^{-1})X_i}{X_i(1 - X_i^2)},$$

$$a_{ij}^- := a_{ij}(X_i^{-1}, X_j),$$

$$b_{ij}^+ := b_{ij}(X_i, X_j^{-1}),$$

$$e^- := e_i(X_i^{-1}).$$

# The Matrix for $A_4$

This is the matrix $A_4$ on the Calogero-Moser subset $\mathrm{CM}_{\delta(\boldsymbol{X})}$ in local coordinates:

$$(A_4)_{ij} = \begin{cases} f_j^- \prod_{\substack{k=1 \\ k-j\neq 0,\pm n}}^{2n} a_{jk}^-, & \text{if } i-j=\pm n \\ f_j^- b_{ij}^+ a_{ij} \prod_{\substack{k=1 \\ k-j\neq 0,\pm n}}^{2n} a_{jk}^-, & \text{if } i-j\neq 0,\pm n \\ t^{2n-2}k_n - \sum_{k\neq j} (A_4)_{ik}, & \text{if } i=j \end{cases},$$

$$a_{ij} = \frac{t^{-1} - tX_iX_j^{-1}}{1 - X_iX_j^{-1}},$$

$$b_{ij} = \frac{t - t^{-1}}{1 - X_iX_j^{-1}},$$

$$f_i = \frac{k_n^{-1} - k_nX_i^2 - (u_n - u_n^{-1})X_i}{1 - X_i^2},$$

$$a_{ij}^- := a_{ij}(X_i^{-1}, X_j),$$

$$b_{ij}^+ := b_{ij}(X_i, X_j^{-1}),$$

$$f^- := f_i(X_i^{-1}).$$

# The Duality Isomorphism

Recall we have $Y_i = T_i \cdots T_{n-1} T_n T_{n-1} \cdots T_1 T_0 T_1^{-1} \cdots T_{i-1}^{-1}$ for $i = 1, ..., n$.

## Proposition 19 ([Sahi], Duality Isomorphism)

*There is a unique involutive algebra isomorphism* $\varepsilon : \mathcal{H}_{n,\mathbf{t},q} \to \mathcal{H}_{n,\widetilde{\mathbf{t}}^{-1},q^{-1}}$ *where*

$$
\begin{aligned}
T_0 &\mapsto S(T_n^\vee)^{-1}S^{-1}, \\
T_i &\mapsto T_i^{-1}, \\
X_i &\mapsto Y_i, \\
q &\mapsto q^{-1}, \\
\mathbf{t} = (k_0, k_n, t, u_0, u_n) &\mapsto (u_n^{-1}, k_n^{-1}, t^{-1}, u_0^{-1}, k_0^{-1}) =: \widetilde{\mathbf{t}}^{-1}.
\end{aligned}
$$

## Remark 20

This induces a corresponding isomorphism $\varepsilon_{\mathrm{CM}}$ on Calogero-Moser spaces.

# Sketching the Main Argument

- We have an isomorphism $\Phi : \mathrm{Spec}(Z_{\delta(\boldsymbol{X})}) \xrightarrow{\sim} \mathrm{CM}_{\delta(\boldsymbol{X})}$.
- But we also have an isomorphism $\varepsilon : Z_{\delta(\boldsymbol{X})} \xrightarrow{\sim} Z_{\delta(\boldsymbol{Y})}$.
- Composing $\varepsilon_{\mathrm{CM}}^{-1} \circ \Phi \circ \varepsilon$ gives us an isomorphism $\mathrm{Spec}(Z_{\delta(\boldsymbol{Y})}) \xrightarrow{\sim} \mathrm{CM}_{\delta(\boldsymbol{Y})}$.
- Use [Oblomkov] to extend $\Phi$ to a regular map on all of $\mathrm{Spec}(Z)$.

Theorem 21 (cf. [Oblomkov])

*The variety* $\mathrm{Spec}(Z)$ *is normal, irreducible and Cohen-Macaulay.*

Thanks for Listening!

# References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