DAHA, Character Varieties and van Diejen's System

Leeds Integrable Systems Seminar Series, University of Leeds

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Goals of the Talk

1. Discuss the following result.

Theorem ([Chalykh–R. '24])

Let $\mathcal{H}$ be the DAHA of type $C^\vee C_n$ at the classical level $q = 1$, $\mathbf{e}$ the Hecke symmetriser and $\mathcal{M}_n$ a certain character variety. Then, we have an isomorphism

$$\mathbf{e}\mathcal{H}\mathbf{e} \cong \mathbb{C}[\mathcal{M}_n].$$

2. Convince you that this is still an integrable systems seminar!

The Double Affine Hecke Algebra of Type $C^\vee C_n$

Definition 1 ([Sahi '99])

Let $q^{1/2}, k_0, k_n, t, u_0, u_n \in \mathbb{C}^*$. The **DAHA of type** $C^\vee C_n$ is the algebra $\mathcal{H}_{q,\boldsymbol{\tau}}$ generated by $T_0^{\pm 1}, T_1^{\pm 1}, \ldots, T_n^{\pm 1}$ and $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ satisfying the following:

$$
\begin{aligned}
[T_i, T_j] &= 0, && |i-j| > 1 \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, && \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && i = 1, \ldots, n-2 \\
T_{n-1} T_n T_{n-1} T_n &= T_n T_{n-1} T_n T_{n-1}, && \\
[X_i, X_j] &= 0, && 1 \le i, j \le n \\
[T_i, X_j] &= 0, && j \ne i, i+1 \\
T_i X_i T_i &= X_{i+1}, && i = 1, \ldots, n-1 \\
(T_0 - k_0)(T_0 + k_0^{-1}) &= 0, && \\
(T_i - t)(T_i + t^{-1}) &= 0, && i = 1, \ldots, n-1 \\
(T_n - k_n)(T_n + k_n^{-1}) &= 0, && \\
(T_0^\vee - u_0)(T_0^\vee + u_0^{-1}) &= 0, && T_0^\vee := q^{-1/2} T_0^{-1} X_1 \\
(T_n^\vee - u_n)(T_n^\vee + u_n^{-1}) &= 0. && T_n^\vee := X_n^{-1} T_n^{-1}
\end{aligned}
$$

PBW and Duality

Throughout, $Y_i := T_i \cdots T_{n-1} T_n T_{n-1} \cdots T_1 T_0 T_1^{-1} \cdots T_{i-1}^{-1}$ and $W = S_n \ltimes \mathbb{Z}_2^n$.

Theorem 2 ([Cherednik '95], PBW Property)

Any $h \in \mathcal{H}_{q,\boldsymbol{\tau}}$ admits a unique presentation of the form

$$h = \sum_{\lambda,\mu \in \mathbb{Z}^n, w \in W} h_{\lambda,w,\mu} \boldsymbol{X}^{\lambda} T_w \boldsymbol{Y}^{\mu}, \qquad h_{\lambda,w,\mu} \in \mathbb{C}.$$

Proposition 3 ([Sahi '99], Duality Isomorphism)

There is a unique involutive algebra isomorphism $\varepsilon : \mathcal{H}_{q,\boldsymbol{\tau}} \to \mathcal{H}_{q^{-1},\boldsymbol{\sigma}}$ where

$$\begin{aligned} T_0 &\mapsto S(T_n^{\vee})^{-1} S^{-1}, \\ T_i &\mapsto T_i^{-1}, \\ X_i &\mapsto Y_i, \\ q &\mapsto q^{-1}, \\ \boldsymbol{\tau} = (k_0, k_n, t, u_0, u_n) &\mapsto (u_n^{-1}, k_n^{-1}, t^{-1}, u_0^{-1}, k_0^{-1}) =: \boldsymbol{\sigma}. \end{aligned}$$

Spherical Subalgebra

Remark 4

Let W be an (affine) Weyl group. We can write any $w \in W$ as a product of simple reflections $w = s_{i1} \cdots s_{i\ell}$, which is **reduced** if ℓ is minimal. We associate to a simple reflection a Hecke generator via $T_i := T_{s_i}$, but this extends to W:

$$T_w := T_{s_i} \cdots T_{s_\ell}.$$

Definition 5

The **spherical subalgebra** of the DAHA is the subalgebra $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$, where

$$\mathbf{e} := \frac{1}{\sum_{w \in W} \tau_w^2} \sum_{w \in W} \tau_w T_w.$$

- When $q = 1$, the centre of the DAHA $\mathcal{Z} \cong \mathbf{e}\mathcal{H}\mathbf{e}$.
- $\mathbf{e}\mathcal{H}\mathbf{e}$ is equipped with a Poisson bracket (q is a deformation parameter).

Definition 6

Fix conjugacy classes $C_1, \ldots, C_k \subseteq \mathrm{GL}_n(\mathbb{C})$, integers $g \geq 0$, $k \geq 1$ and define

$$\mathfrak{R}_{g,k} := \{X_1, Y_1, \ldots, X_g, Y_g \in \mathrm{GL}_n(\mathbb{C}), A_i \in C_i : X_1 Y_1 X_1^{-1} Y_1^{-1} \cdots X_g Y_g X_g^{-1} Y_g^{-1} A_1 \cdots A_k = \mathbb{1}_n\}.$$

The corresponding $\mathrm{GL}_n(\mathbb{C})$-**character variety** is the variety of closed orbits:

$$\mathfrak{M}_{g,k} := \mathfrak{R}_{g,k} /\!\!/ \mathrm{GL}_n(\mathbb{C}).$$

Remark 7

In other words, points of the character variety are isomorphism classes of representations of the fundamental group $\pi_1(\Sigma_{g,k})$ by matrices in $\mathrm{GL}_n(\mathbb{C})$.

Proposition 8 ([Hausel–Letellier–Rodriguez-Villegas '13])

For "generic" semi-simple C_i, the varieties $\mathfrak{M}_{g,k}$ are smooth and connected.

Calogero-Moser Space in Type $C^\vee C_n$

Let $g = 0$ and $k = 4$, so we are working on the four-punctured sphere $\Sigma_{0,4}$.

$$C_1 = [\mathrm{diag}(\underbrace{-k_0^{-1}, \ldots, -k_0^{-1}}_{n}, \underbrace{k_0, \ldots, k_0}_{n})],$$

$$C_2 = [\mathrm{diag}(\underbrace{-u_0^{-1}, \ldots, -u_0^{-1}}_{n}, \underbrace{u_0, \ldots, u_0}_{n})],$$

$$C_3 = [\mathrm{diag}(\underbrace{-u_n^{-1}, \ldots, -u_n^{-1}}_{n}, \underbrace{u_n, \ldots, u_n}_{n})],$$

$$C_4 = [\mathrm{diag}(\underbrace{-k_n^{-1}, \ldots, -k_n^{-1}}_{n}, \underbrace{k_n t^{-2}, \ldots, k_n t^{-2}}_{n-1}, \underbrace{k_n t^{2n-2}}_{1})].$$

Definition 9

The **Calogero-Moser space** is the character variety $\mathfrak{M}_{0,4}$ with the above data:

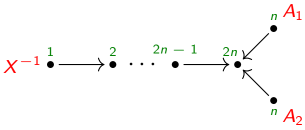
$$\mathcal{M}_n := \{A_i \in C_i : A_1 A_2 A_3 A_4 = \mathbb{1}_{2n}\} / \mathrm{GL}_{2n}(\mathbb{C}).$$

There is a correspondence between these matrices and DAHA elements as follows:

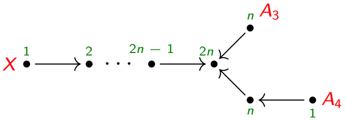
$$A_1 \leftrightarrow q^{1/2} T_0, \qquad A_2 \leftrightarrow T_0^\vee, \qquad A_3 \leftrightarrow S T_n^\vee S^{-1}, \qquad A_4 \leftrightarrow S T_n S^\dagger.$$

Local Coordinates

Solving $A_1 A_2 A_3 A_4 = \mathbb{1}_{2n}$ where $A_i \in C_i$ can be split into two related problems.



(a) The quiver Q_1 for $A_1 A_2 X^{-1} = \mathbb{1}_{2n}$.



(b) The quiver Q_2 for $X A_3 A_4 = \mathbb{1}_{2n}$.

This introduces a coordinate chart on $\mathcal{M}_n$. Solving the problem encoded by Q_1 is easy, but the problem encoded by Q_2 is non-Dynkin and more difficult. We use the theory from [Crawley-Boevey–Shaw '06] when working with these quivers.

Lemma 10

Any solution to the Q_1-problem is isomorphic to a direct sum of 2×2 solutions.

Local Coordinates

From the DAHA, $A_1A_2 \leftrightarrow q^{1/2}T_0T_0^\vee = X_1$. From the point-of-view of matrices,

$$A_1A_2 = \mathrm{diag}(X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}).$$

Let A_1°, A_2° be arbitrary 2×2 matrices with $A_1^\circ A_2^\circ = \mathrm{diag}(X_1, X_1^{-1})$ and $A_i^\circ \in C_i^\circ$ (the conjugacy classes defining $\mathcal{M}_n$ but in the 2×2 case). Explicitly, we solve

$$\begin{cases} ae + bg = X_1, & ad - bc = -1, \\ af + bh = 0, & eh - fg = -1, \\ ce + dg = 0, & a + d = k_0 - k_0^{-1}, \\ cf + dh = X_1^{-1}, & e + h = u_0 - u_0^{-1}. \end{cases}$$

We obtain n coordinates $\boldsymbol{X}$ by solving this matrix problem and taking the direct sum. Another n coordinates $\boldsymbol{P}$ encode how to glue the two problems together.

The Matrix A_1

Consider the rational functions

$$a_i = \frac{k_0^{-1} - qk_0X_i^{-2} - q^{1/2}(u_0 - u_0^{-1})X_i^{-1}}{1 - X_i^{-2}},$$

$$b_i = \frac{(k_0 - k_0^{-1}) + q^{1/2}(u_0 - u_0^{-1})X_i^{-1}}{1 - qX_i^{-2}}.$$

The matrix A_1 in the local Calogero-Moser coordinates $(\boldsymbol{X}, \boldsymbol{P})$ is

$$(A_1)_{ij} = \begin{cases} b_i & \text{if } i = j \\ a_i P_i^{-1} & \text{if } i - j = \pm n \ , \\ 0 & \text{otherwise} \end{cases}$$

The Matrix A_2

Consider the rational functions

$$a_i = \frac{k_0^{-1} - qk_0 X_i^{-2} - q^{1/2}(u_0 - u_0^{-1})X_i^{-1}}{1 - X_i^{-2}},$$

$$b_i = \frac{(k_0 - k_0^{-1}) + q^{1/2}(u_0 - u_0^{-1})X_i^{-1}}{1 - qX_i^{-2}}.$$

The matrix A_2 in the local Calogero-Moser coordinates $(\boldsymbol{X}, \boldsymbol{P})$ is

$$(A_2)_{ij} = \begin{cases} q^{-1/2}\left(b_i - (k_0 - k_0^{-1})\right) X_i & \text{if } i = j \\ q^{-1/2} a_i P_i^{-1} X_i^{-1} & \text{if } i - j = \pm n \\ 0 & \text{otherwise} \end{cases},$$

The Matrix A_3

The matrix A_3 in the local Calogero-Moser coordinates $(\boldsymbol{X}, \boldsymbol{P})$ is

$$(A_3)_{ij} = \begin{cases} X_i^{-1} c_j^- \prod_{k=1}^{2n} {}^{\diamond}\, a_{ik}^- & \text{if } i - j = \pm n \\ -X_i^{-1} c_j^- b_{ij}^- a_{ij} \prod_{k=1}^{2n} {}^{\diamond}\, a_{jk}^- & \text{if } i - j \neq 0, \pm n \\ X_i^{-1} k_n^{-1} t^{2-2n} - \sum_{k \neq i} (A_3)_{ik} & \text{if } i = j \end{cases},$$

$$a_{ij} = \frac{t^{-1} - tX_i X_j^{-1}}{1 - X_i X_j^{-1}},$$

$$b_{ij} = \frac{t - t^{-1}}{1 - X_i X_j^{-1}},$$

$$c_j = \frac{k_n^{-1} - k_n X_j^2 - (u_n - u_n^{-1}) X_j}{1 - X_j^2},$$

$$a_{ij}^- := a_{ij}(X_i^{-1}, X_j),$$

$$b_{ij}^+ := b_{ij}(X_i, X_j^{-1}),$$

$$c_j^- := c_j(X_j^{-1}).$$

The Matrix A_4

The matrix A_4 in the local Calogero-Moser coordinates $(\boldsymbol{X}, \boldsymbol{P})$ is

$$(A_4)_{ij} = \begin{cases} d_j^- \prod_{k=1}^{2n}{}^{\diamond} a_{jk}^- & \text{if } i-j = \pm n \\ d_j^- b_{ij}^+ a_{ij} \prod_{k=1}^{2n}{}^{\diamond} a_{jk}^- & \text{if } i-j \neq 0, \pm n \\ k_n t^{2n-2} - \sum_{k \neq j} (A_4)_{ik} & \text{if } i = j \end{cases},$$

$$a_{ij} = \frac{t^{-1} - tX_iX_j^{-1}}{1 - X_iX_j^{-1}},$$

$$b_{ij} = \frac{t - t^{-1}}{1 - X_iX_j^{-1}},$$

$$d_i = \frac{k_n^{-1} - k_nX_j^2 - (u_n - u_n^{-1})X_j}{1 - X_j^2},$$

$$a_{ij}^- := a_{ij}(X_i^{-1}, X_j),$$

$$b_{ij}^+ := b_{ij}(X_i, X_j^{-1}),$$

$$d_j^- := f_j(X_j^{-1}).$$

More Local Coordinates

The Duality Isomorphism ε induces an isomorphism on character varieties:

$$\mathcal{E} : \mathcal{M}_n \ni (A_1, A_2, A_3, A_4) \mapsto (A_3^{-1}, A_2^{-1}, A_1^{-1}, A_4^{-1}) \in \mathcal{M}'_n.$$

For another chart on $\mathcal{M}_n$, work on the dual $\mathcal{M}'_n$ and transfer coordinates over:

- $(\boldsymbol{X}, \boldsymbol{P})$ **coordinates:** A_1, A_2 are straightforward; A_3, A_4 are complicated.
- $(\boldsymbol{Y}, \boldsymbol{Q})$ **coordinates:** A_3, A_2 are straightforward; A_1, A_4 are complicated.

Remark 11

The proof of the main result is now possible by adapting [Oblomkov '04].

Poisson Bracket

Proposition 12

The $(\boldsymbol{X}, \boldsymbol{P})$ *coordinates on* $\mathcal{M}_n$ *are log-canonical, i.e.*

$$\{X_i, X_j\} = \{P_i, P_j\} = 0, \qquad \{P_i, X_j\} = \delta_{ij} P_i X_j.$$

Proof

Embed the DAHA $\mathcal{H}_{q,\boldsymbol{\tau}} \hookrightarrow \mathbb{C}_q(\boldsymbol{X})[\boldsymbol{P}^{\pm 1}]$ (Basic Representation). The brackets follow by viewing the coordinates P_i, X_j in this quantised algebra, with relations

$$[X_i, X_j] = [P_i, P_j] = 0, \qquad P_i X_j = q^{\delta_{ij}} X_j P_i.$$

□

Lemma 13

The $h_k := \operatorname{tr} X^k$ *are in Poisson involution, and for each* $i = 1, \ldots, n$, *we have*

$$\{X_i, h_k\} = 0, \qquad \{P_i, h_k\} = k P_i (X_i^k - X_i^{-k}).$$

Hamiltonian Dynamics

The Hamiltonian dynamics governed by h_k in $(\boldsymbol{X}, \boldsymbol{P})$ coordinates is separated:

$$X_i(t) = X_i(0), \qquad P_i(t) = e^{kt(X_i^k - X_i^{-k})} P_i(0).$$

Proposition 14

The Hamiltonian dynamics on $\mathcal{M}_n$ governed by $h_k = \operatorname{tr} X^k$ can be obtained by projecting the following dynamics from the pre-quotient $\mathfrak{R}_{0,4}$ onto $\mathcal{M}_n$:

$$\dot{A}_1 = -k(A_1 X^k - X^k A_1), \qquad \dot{A}_2 = -k(A_2 X^k - X^k A_2), \qquad \dot{A}_3 = \dot{A}_4 = 0.$$

The above dynamics integrate, giving $X = A_1 A_2$ constant and

$$A_1(t) = e^{ktX^k} A_1(0) e^{-ktX^{-k}}, \qquad A_3(t) = A_3(0),$$
$$A_2(t) = e^{ktX^k} A_2(0) e^{-ktX^{-k}}, \qquad A_4(t) = A_4(0).$$

Dual Hamiltonian Dynamics

The Duality Isomorphism $\mathcal{E}$ on the character variety interchanges $X = A_1 A_2$ with $Y = A_4 A_1$, a Lax matrix for the van Diejen system [Chalykh '19, Corollary 4.4].

Proposition 15

The Hamiltonian dynamics on $\mathcal{M}_n$ governed by $H_k := \operatorname{tr} Y^k$ can be obtained by projecting the following dynamics from the pre-quotient $\mathfrak{R}_{0,4}$ onto $\mathcal{M}_n$:

$$\dot{A}_1 = \dot{A}_4 = 0, \qquad \dot{A}_2 = -k(Y^k A_2 - A_2 Y^k), \qquad \dot{A}_3 = -k(Y^k A_3 - A_3 Y^k).$$

The above dynamics integrate, giving $Y = A_4 A_1$ constant and

$$A_1(t) = A_1(0), \qquad A_2(t) = e^{-ktY^{-k}} A_2(0) e^{ktY^k},$$
$$A_4(t) = A_4(0), \qquad A_3(t) = e^{-ktY^{-k}} A_3(0) e^{ktY^k}.$$

van Diejen Hamiltonians

In dual coordinates, we see that

$$X(t) = A_1(t)A_2(t) = A_1(0)e^{-ktY^k}A_2(0)e^{ktY^k} = X(0)e^{kt(Y^k - Y^{-k})}$$

Remark 16

The two-parameter case $k_0 = u_0 = u_n = 1$, well-studied by [Pusztai–Görbe '17], complements our setting in that the above formula matches (4.113) in *op. cit.*.

We have two integrable systems: one dictated by h_k, and another by H_k.

Remark 17

The dual chart on $\mathcal{M}_n$ provides the action-angle variables for the van Diejen system. The action variables are the eigenvalues of Y, and the angle variables are the dual counterparts of the P_i.

Writing the first set of Hamiltonians $h_k = \operatorname{tr} X^k$ in terms of these action-angle coordinates, one obtains the van Diejen Hamiltonians in dual parameters. This acts as an analogue to Ruijsenaars duality.

Space of Graph Connections

Fock and Rosly explain how to obtain $\mathcal{M}_n$ via Hamiltonian reduction by considering combinatorial connections on embedded graphs (so-called graph connections).

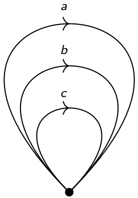


Figure: The graph ℓ corresponding to $\mathcal{M}_n$.

General Set-up: The space of graph connections is $\mathcal{A}^\ell = \prod G$, so we attach to each edge a group element, and we quotient out by the natural action of the gauge group $\mathcal{G}^\ell$.

Our Set-up: For the above graph, we have $\mathcal{A}^\ell = \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})$, with the gauge group $\mathcal{G}^\ell = \mathrm{GL}_{2n}(\mathbb{C})$ acting on $\mathcal{A}^\ell$ by simultaneous conjugation.

Fock–Rosly Bracket

Due to [Fock–Rosly '99], the Poisson structure on $\mathcal{A}^\ell$ is dictated by choosing a classical r-matrix at each vertex. In our set-up, we make the standard choice

$$r = \sum_{i<j}^{2n} E_{ij} \otimes E_{ji} + \frac{1}{2} \sum_{i=1}^{2n} E_{ii} \otimes E_{ii}.$$

Notation 18

The respective (skew-)symmetric parts are $r_a = \frac{1}{2}(r - r_{21})$ and $t = \frac{1}{2}(r + r_{21})$.

Let $\mathbf{A}$ be the matrix representing the edge a, and so forth. Then, we have

$$\mathbf{A} \sim A_1, \qquad \mathbf{A}^{-1}\mathbf{B} \sim A_2, \qquad \mathbf{B}^{-1}\mathbf{C} \sim A_3, \qquad \mathbf{C}^{-1} \sim A_4.$$

Lemma 19

The Poisson brackets on $\mathcal{A}^\ell$ are (all given analogously to)

$$\{\mathbf{A}, \mathbf{A}\} = r_a(\mathbf{A} \otimes \mathbf{A}) + (\mathbf{A} \otimes \mathbf{A})r_a + (\mathbb{1} \otimes \mathbf{A})r_{21}(\mathbf{A} \otimes \mathbb{1}) - (\mathbf{A} \otimes \mathbb{1})r(\mathbb{1} \otimes \mathbf{A}),$$
$$\{\mathbf{A}, \mathbf{B}\} = r(\mathbf{A} \otimes \mathbf{B}) - (\mathbf{A} \otimes \mathbf{B})r_{21} + (\mathbb{1} \otimes \mathbf{B})r_{21}(\mathbf{A} \otimes \mathbb{1}) - (\mathbf{A} \otimes \mathbb{1})r(\mathbb{1} \otimes \mathbf{B}).$$

Application to the Main Result

Proposition 20

The main isomorphism $\mathbf{e}\mathcal{H}\mathbf{e} \cong \mathbb{C}[\mathcal{M}_n]$ *is a Poisson map, i.e. it identifies the natural Poisson bracket on the spherical subalgebra with the Fock–Rosly bracket on the character variety.*

Corollary 21

The $(\boldsymbol{X}, \boldsymbol{P})$ *coordinates on* $\mathcal{M}_n$ *are log-canonical with respect to the Fock–Rosly bracket, and the spherical subalgebra provides a quantisation of the character variety.*

Remark 22

Let $\mathcal{M}$ be the moduli space of flat $\mathrm{GL}_{2n}(\mathbb{C})$-connections on the four-punctured sphere $\Sigma_{0,4}$. Then, $\mathcal{M}_n$ is a symplectic leaf (by specifying the conjugacy classes C_i) and we can view it as a completed phase space for the van Diejen system.

Alternative Description

Proposition 23

$\mathcal{M}_n \cong \mathcal{R}_n / \mathrm{GL}_{2n}(\mathbb{C})$, where $\mathcal{R}_n$ is the set of $X, Y, T \in \mathrm{GL}_{2n}(\mathbb{C})$ subject to

$$\begin{aligned} T - T^{-1} &= (u_0 - u_0^{-1})\mathbb{1}_V, \\ XT^{-1} - TX^{-1} &= (k_0 - k_0^{-1})\mathbb{1}_V, \\ T^{-1}Y^{-1} - YT &= (u_n - u_n^{-1})\mathbb{1}_V, \\ tYTX^{-1} - t^{-1}XT^{-1}Y^{-1} &= (k_n t^{-1} - k_n^{-1} t)\mathbb{1}_V + (t - t^{-1})vw, \\ wv &= \frac{t^{2n} - 1}{t^2 - 1} k_n + \frac{1 - t^{-2n}}{1 - t^{-2}} k_n^{-1}. \end{aligned}$$

Lemma 24 ([Chalykh '19])

Let $\mathbf{v} = (1, \ldots, 1)^T$ and $\mathbf{w} = (\phi_1, \ldots, \phi_{2n})$, where $\phi_i := c_i^- \prod_{k \neq i}^{n} a_{ki} a_{ki}^-$. The Hecke symmetriser acts by a constant multiple of the rank-one matrix $\mathbf{vw}$.

Quantisation

For $q = 1$, the ring of functions $\mathbb{C}[\mathcal{M}_n]$ is generated by traces of words in A_i.

Lemma 25

The algebra $\mathbb{C}[\mathcal{M}_n]$ is generated by $wa(X, Y, T)v$ for $a \in \mathbb{C}\left\langle X^{\pm 1}, Y^{\pm 1}, T^{\pm 1}\right\rangle$.

Notation 26

Let $\mathscr{D}_q = \mathbb{C}_q(\boldsymbol{X})[\boldsymbol{P}^{\pm 1}] \rtimes \mathbb{C}W$ be the ring of q-difference-reflection operators.

Proposition 27

The elements $\mathbf{w}a(X, Y, T)\mathbf{v}$ generate a subalgebra of $\mathscr{D}_q^W$ isomorphic to the spherical subalgebra $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$, for $a \in \mathbb{C}\left\langle X^{\pm 1}, Y^{\pm 1}, T^{\pm 1}\right\rangle$.

Corollary 28

As an algebra, $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$ is generated by $\mathbf{e}X_1^n Y_1^m \mathbf{e}$ and $\mathbf{e}X_1^n Y_1^m T_0^\vee \mathbf{e}$ for $n, m \in \mathbb{Z}$.

Future Work

- The conjecture of [Etingof–Gan–Oblomkov '06] that we prove is actually only part of a more general statement about *generalised* DAHAs with underlying star-shaped quiver Q. We have settled the case $Q = \widetilde{D}_4$, but the cases $Q = \widetilde{E}_{6,7,8}$ remain open.

- Using the interpretation as a moduli space of flat connections, this leads to a mapping class group action. This suggests the study of a spin version which would yield a new integrable system.

- Recent work [Braverman–Finkelberg–Nakajima '19] on quantised Coulomb branches of $3d$ $\mathcal{N} = 4$ gauge theories suggests a further generalisation to $\widetilde{D}_m$ where $m > 4$.

Thanks for Listening!

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