Character Varieties and Symmetric Polynomials

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Character Varieties

Recall that the *fundamental group* of a topological space X based at $x \in X$ is the group of homotopy classes $\pi_1(X, x) = \{[\alpha] : \alpha \text{ is a loop based at } x\}$.

Definition 1

A **character variety** $\mathcal{M}$ is a space whose points are isomorphism classes of representations in $\mathrm{GL}_m(\mathbb{C})$ of the fundamental group of a Riemann surface with genus g and k punctures.

Example 2 (Our Situation)

Consider the punctured Riemann sphere $\mathbb{CP}^1 \setminus \{\text{four points}\}$: $g = 0$ and $k = 4$. The fundamental group is generated by four loops, one around each puncture.

The character variety we care for arises by assigning some 2×2 matrix to each loop, living in a prescribed conjugacy class. The product of matrices is the identity $\mathbb{1}_2$. The multiplicities μ_{ij} of the four pairs of eigenvalues are given in

$$\mu_1 = (1, 1), \qquad \mu_2 = (1, 1), \qquad \mu_3 = (1, 1), \qquad \mu_4 = (1, 1).$$

Remark 3

In order to avoid singularities in $\mathcal{M}$, we choose our eigenvalues to be generic.

We are interested in topological properties of $\mathcal{M}$ (e.g. dimension, connectedness).

Proposition 4 ([Hausel–Letellier–Rodriguez-Villegas '11, Theorem 2.1.5])

Let μ_{ij} be the multiplicities of the prescribed conjugacy classes at the i^{th} puncture. If non-empty, $\mathcal{M}$ is a smooth variety of dimension

$$d = (2g - 2 + k)m^2 - \sum_{i,j} \mu_{ij}^2 + 2.$$

Example 5 (Our Situation)

Here, $g = 0$, $k = 4$ and our matrices have sizes $m = 2$. Substituting these along with $\mu_{ij} = 1$ for all i, j into the formula above produces

$$d = (0 - 2 + 4)2^2 - 8 + 2 = 2.$$

Recall that the *Poincaré polynomial* of a topological space is the generating function of its Betti numbers (dimensions of homology groups). Rather recently, Anton Mellit proved a formula for the Poincaré polynomial of $\mathcal{M}$ (shown later).

Remark 6

The zeroth Betti number is the number of connected components of the space.

Young Tableaux

Definition 7

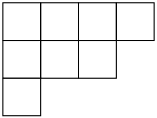
A **partition** of an integer m into ℓ parts is a sequence of positive integers $\mu = (\mu_1, ..., \mu_\ell)$ where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell$. The corresponding **Young diagram** is a collection of boxes such that there are ℓ rows enumerated $1, ..., \ell$ from top-to-bottom and the i^{th} row has μ_i boxes (with no gaps between boxes).

Notation 8

The set of partitions of an integer m is denoted $\mathcal{P}_m$.

Example 9

Consider the partition $\mu = (4, 3, 1)$. The Young diagram of μ is as follows:



Definition 10

A **Young tableaux** of shape $\mu \in \mathcal{P}_m$ is a filling of the Young boxes by some positive integers, i.e. we have an assignment $\phi : \mu \to \mathbb{Z}^+$ called a **filling** of μ.

Young Tableaux

Definition 11 ([Haglund–Haiman–Loehr '05, cf. (13)])

Let ϕ be a filling of $\mu \in \mathcal{P}_m$. A **descent of** ϕ is a Young box where the filling of the cell immediately to the left is *strictly* smaller than its own filling. The set of descents is denoted $\mathrm{Des}(\phi)$.The **major statistic** of the filling ϕ is defined as

$$\mathrm{maj}(\phi) := |\mathrm{Des}(\phi)| + \sum_{u \in \mathrm{Des}(\phi)} a(u),$$

where the **arm** $a(u)$ is the number of cells in the same row but to the right of u.

Example 12

Consider the partition $\mu = (4, 3, 1)$ with the filling ϕ as given below:

1	7	7	5
2	4	6	
3			

$$\mathrm{maj}(\phi) = 3 + 2 + 1 + 0.$$

Remark 13

There is another number associated to a filling ϕ called the inversion statistic $\mathrm{inv}(\phi)$. This isn't too hard to calculate but the rules of the game here are more complicated. We don't need it, but we may state its values where necessary.

Symmetric Polynomials

Definition 14

A function is **symmetric** if it is invariant under interchanging variables: for k variables and any $\sigma \in S_k$, we have $f(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = f(X_1, \ldots, X_k)$.

Example 15

A homogeneous symmetric *polynomial* is the sum of all monomials of a fixed total degree. For example, we have $h_3(X_1, X_2) = X_1^3 + X_1X_2^2 + X_1^2X_2 + X_2^3$.

Let Λ be the algebra of symmetric functions in infinitely-many variables $\mathbf{X} = (X_i)$ with coefficients in the field of rational functions $\mathbb{Q}(q, t)$. A special basis of Λ is that consisting of the (*transformed*) *Macdonald polynomials* $\widetilde{H}_\mu[\mathbf{X}; q, t]$.

Theorem 16 ([HHL05, Theorem 7.12])

Let $\mu \in \mathcal{P}_m$ and ϕ denote a filling. The Macdonald polynomials are given by

$$\widetilde{H}_\mu[\mathbf{X}; q, t] = \sum_{\phi} q^{\mathrm{maj}(\phi)} t^{\mathrm{inv}(\phi)} \mathbf{X}^\phi, \qquad \mathbf{X}^\phi := \prod_{u \in \mu} X_{\phi(u)}.$$

Remark 17

There is a symmetry $\widetilde{H}_\mu[\mathbf{X}; q, t] = \widetilde{H}_{\mu'}[\mathbf{X}; t, q]$, where μ' is the transpose of μ.

Symmetric Polynomials

Example 18

We consider fillings of $\mu = (2) = \boxed{}\boxed{}$ in order to compute $\widetilde{H}_{(2)}[\mathbf{X}; q, t]$:

(i) Let $\phi = 1^2$; we fill with two 1s. There is only one way to do this:

$$\boxed{1}\boxed{1} \quad \text{with maj} = 0 \text{ (and inv} = 0).$$

(ii) Let $\phi = 1^1 2^1$; we fill with one 1 and one 2. There are two ways to do this:

$$\boxed{1}\boxed{2} \quad \text{with maj} = 1 \text{ (and inv} = 0),$$

$$\boxed{2}\boxed{1} \quad \text{with maj} = 0 \text{ (and inv} = 0).$$

(iii) Let $\phi = 2^2$; we fill with two 2s. There is only one way to do this:

$$\boxed{2}\boxed{2} \quad \text{with maj} = 0 \text{ (and inv} = 0).$$

We also see that (i) $\mathbf{X}^\phi = X_1^2$, (ii) $\mathbf{X}^\phi = X_1 X_2$ and (iii) $\mathbf{X}^\phi = X_2^2$. Therefore,

$$\widetilde{H}_{(2)}[\mathbf{X}; q, t] = X_1^2 + (q+1)X_1X_2 + X_2^2.$$

Remark 19

In *monomial symmetric polynomials*, $\widetilde{H}_{(2)}[\mathbf{X}; q, t] = m_{(2)}[\mathbf{X}] + (q+1)m_{(1,1)}[\mathbf{X}]$.

Connectedness

Theorem 20 ([Mellit '17, Theorem 7.12])

The Poincaré polynomial of the character variety $\mathcal{M}$ is given by

$$P(\mathcal{M}, q) = q^{d/2} \prod_{i=1}^{k} \left\langle \mathbb{H}_{g,k}^{\mathrm{HLV}}[\mathbf{X}; T, q^{-1}, 1]\Big|_{T^m}, \prod_j h_{\mu_{ij}}[\mathbf{X}] \right\rangle$$

where μ_{ij} are the eigenvalue multiplicities in the prescribed conjugacy classes.

- The d is the dimension of $\mathcal{M}$ from Proposition 4.
- The $\mathbb{H}_{g,k}^{\mathrm{HLV}}$ is an explicit generating series involving Macdonald polynomials.
- The $\Big|_{T^m}$ tells us to look only at the T^m-coefficient in the above series.
- The $\langle -, - \rangle$ is an inner product on Λ in which $m_\lambda[\mathbf{X}]$ and $h_\mu[\mathbf{X}]$ are dual.
- This duality restricts the fillings ϕ we consider when using Theorem 16.

Example 21 (Our Situation)

We omit more technicalities but Theorem 20 will give us $P(\mathcal{M}, q) = 1 + 5q$.

Why I Care

- I care about $\mathcal{M}$ with generic eigenvalues and multiplicities

$$(n, n), \qquad (n, n), \qquad (n, n), \qquad (n, n-1, 1).$$

- The goal of my research project is to establish a link between the above character variety and a double affine Hecke algebra; this settles a conjecture in [Etingof–Gan–Oblomkov '06].
- Connectedness of $\mathcal{M}$ is an important step towards this goal.
- The full polynomial $P(\mathcal{M}, q)$ is obtainable conjecturally through other means but this is something to study later.
- There is some relationship to quiver varieties (our story is associated to the *framed* $\widetilde{D}_4$ quiver). There are similar character varieties for framed $\widetilde{E}_{6,7,8}$ whose connectedness is also provable using the combinatorics here.

Thanks for Listening!

References

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