A Deligne-Simpson Problem

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The Problem

We consider this **multiplicative Deligne-Simpson problem**: for given conjugacy classes $C_1, \dots, C_k$ in $\mathrm{GL}_n(\mathbb{C})$, determine irreducible solutions of the equation

$$A_1 \cdots A_k = \mathbb{1}_n, \qquad \text{where } A_i \in C_i.$$

Remark 1

The term *irreducible* above means there is no common proper invariant subspace under the action of the matrices A_i. If we omit this condition, the problem will be harder.

Example 2 (Our Situation)

We want to solve $A_1A_2A_3A_4 = \mathbb{1}_{2n}$, where each matrix is of size $2n \times 2n$ with prescribed eigenvalues (i.e. they are each diagonalisable).

We focus on the general case for now.

Monodromy

Monodromy is the study of an object's behaviour near a singularity (apparently, the word comes from Greek and means 'uniformly running').

Definition 3

Let $A_1, ..., A_k$ be $n \times n$ matrices. A **Fuchsian system** is a linear system

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \left(\sum_{i=1}^{k} \frac{A_i}{\lambda - t_i} \right) \Phi,$$

where $\lambda \in \mathbb{CP}^1 \setminus \{t_1, ..., t_k\}$ and Φ is an $n \times n$ matrix of dependent variables.

Remark 4

Recall that $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, the one-point compactification of the complex plane. In Definition 3, if we impose the **additive** Deligne Simpson condition

$$A_1 + \cdots + A_k = 0,$$

then the point at infinity $\lambda = \infty$ is a so-called **regular point** of the system; any neighbourhood of this point contains a solution of the Fuchsian system with the condition that $\Phi|_{\lambda=\infty} = \mathbb{1}_n$.

Monodromy

Notation 5

Throughout, let Σ_k be the k-punctured Riemann sphere $\mathbb{CP}^1 \setminus \{t_1, \ldots, t_k\}$.

We now define a group associated with the punctured Riemann sphere. To do this, we first fix a base-point $x \in \Sigma_k$ and a matrix $B \in \mathrm{GL}_n(\mathbb{C})$.

Definition 6

Let γ be a loop at $x \in \Sigma_k$ which encircles the punctures. The corresponding **monodromy operator** is a linear operator $M : \Phi \mapsto \Phi_\gamma$, where $\Phi|_{\lambda=x} = B$ is a solution of the Fuchsian system and Φ_γ is its analytic continuation along γ.

Remark 7

This is a bit technical already so note the punchline: the **monodromy group** is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ generated by these monodromy operators M. Essentially, the group is understood as a representation (homomorphism) of the form

$$\rho : \pi_1(\Sigma, x) \to \mathrm{GL}_n(\mathbb{C}).$$

- $\pi_1(\Sigma, x)$ is generated by loops γ_i at x, where γ_i encircles the puncture t_i.
- The representation assigns to each generator a **monodromy matrix** M_i.
- Hence, the concatenation of paths $\gamma_k \cdots \gamma_1 \simeq 0$ tells us $M_1 \cdots M_k = \mathbb{1}_n$.

Monodromy

The headline is the following:

$$\textbf{Multiplicative Deligne-Simpson} \xleftrightarrow{\text{monodromy}} \textbf{Additive Deligne-Simpson}.$$

Example 8

Let A_1, A_2, A_3 be 2×2 matrices and consider the 4-punctured Riemann sphere $\Sigma := \mathbb{CP}^1 \setminus \{0, 1, t, \infty\}$. The Fuchsian system of interest is the following:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1} + \frac{A_3}{\lambda - t} \right) \Phi,$$

where we define the matrix corresponding to infinity as $A_\infty := -A_1 - A_2 - A_3$.

Theorem 9 ([CM10, (5.30, 5.31)])

The $M_i \in \mathrm{GL}_2(\mathbb{C})$ *corresponding to* Example 8 *are given locally as*

$$M_i \sim \exp(A_i) \sim \begin{pmatrix} e^{\theta_i/2} & 0 \\ 0 & -e^{-\theta_i/2} \end{pmatrix},$$

where $\pm\theta_i/2$ *are the eigenvalues of the matrix* A_i*.*

Other Areas

My research has concerned the double affine Hecke algebra of type BC. Through this, we can obtain a family of solutions to the multiplicative Deligne-Simpson problem described at the beginning as a by-product.

- The *generalised* double affine Hecke algebra in [EGO06] has the relation

$$U_1 U_2 U_3 U_4 S S^{\dagger} = 1,$$

  and there is an injection from this to 'our' double affine Hecke algebra.

- Using methods in the paper [Cha19], I was able to construct an explicit $2n$-parameter family of matrices A_1, A_2, A_3, A_4 and prove that they have prescribed eigenvalues – straightforward for A_1, A_2, A_3 and harder for A_4.

- The eigendata can define a representation of a star-like quiver (or rather the multiplicative quiver variety) in the spirit of [CBS04].

- Lately, we interpret the $2n$ parameters as defining a chart on an algebraic variety. Proving that said variety is connected boils down to the cohomology of Σ_4, information obtainable via [Mel17, Theorem 7.12].

Thanks for Listening!

References

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