Double Affine Hecke Algebras and Character Varieties

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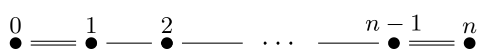
Ivan Cherednik [Che92] introduced the double affine Hecke algebra (DAHA) associated to a reduced irreducible root system in order to prove conjectures regarding Macdonald polynomials. Subsequent work by Masatoshi Noumi [Nou95] and Siddhartha Sahi [Sah99] extended the theory to cover the setting of an arbitrary irreducible *affine* root system. In particular, the non-reduced irreducible affine root system of type $C^\vee C_n$ contains, as subsystems, all non-reduced irreducible affine root systems of classical type; this is noted in Ian Macdonald's book [Mac03, p.12].

Slide 2 – Hecke Algebras of Type C_n

Definition The Weyl group of type C_n is generated by $s_1, ..., s_n$ which satisfy the quadratic relation $s_i^2 = 1$ for all $i = 1, ..., n$ and satisfy the braid relations $s_i s_j s_i \cdots = s_j s_i s_j \cdots$ where there are $\mathrm{ord}(s_i s_j)$-many terms on each side. The affine Weyl group of type $\widetilde{C}_n$ is generated by an additional element s_0 which satisfies the quadratic relation, as well as $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

The affine Coxeter diagram of type $\widetilde{C}_n$ encodes the braid relations in the affine Weyl group: $\mathrm{ord}(s_i s_j) = 2, 3, 4$ says the i^{th} and j^{th} nodes are connected by zero/one/two edges respectively.



Remark We can think of the quadratic relations as being deformed into the Hecke relations:

$$s_i^2 = 1 \quad \Leftrightarrow \quad (s_i - 1)(s_i + 1) = 0 \qquad \rightsquigarrow \qquad (T_i - \tau_i)(T_i + \tau_i^{-1}) = 0.$$

Slide 3 – Generators of the Affine Hecke Algebra

Approach 1: an affine Hecke algebra associated to the affine Weyl group $\widetilde{W}$ has basis $\{T_w\}_{w \in \widetilde{W}}$. The idea is to write any affine Weyl element $w = s_{i_1} \cdots s_{i_k}$ as a product of simple reflections in a *reduced* (minimal) way. Having defined $T_i := T_{s_i}$, this can be extended to $T_w := T_{i_1} \cdots T_{i_k}$.

Approach 2: the affine Weyl group $\widetilde{W} \cong W \ltimes \tau(\mathbb{Z}^n)$, where $\tau(\mathbb{Z}^n)$ are translations in an integer lattice (really the coroot lattice $Q^\vee$ associated with the root system of type C_n). The Y_i are now Hecke elements corresponding to a translation in the direction of ε_i.

Slide 4 – The DAHA of Type $C^\vee C_n$

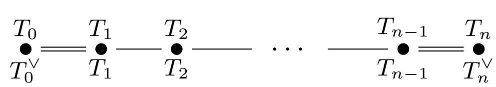
We can clearly see the finite Hecke algebra of type $C^\vee C_n$ inside Definition 5 with the quoted braid relations, and an obvious copy of one affine Hecke algebra of type $\widetilde{C}_n$ sitting inside. However, the second affine Hecke algebra is a little more obscure. We now justify why the DAHA contains two copies of an affine Hecke algebra that overlap on a finite Hecke algebra.

We can use the theorem of Lusztig to obtain a linear basis of the DAHA. Indeed, the previous result tells us that X_i pairwise commute and generate a subalgebra $\mathbb{C}[\boldsymbol{X}^{\pm 1}]$. Hence, since the DAHA is the affine Hecke algebra with this extra collection of elements, we have this vector space isomorphism where $\widetilde{H}$ is the affine Hecke algebra:

$$\mathcal{H} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}] \otimes \widetilde{H}.$$

Corollary *As vector spaces, we have* $\mathcal{H} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}] \otimes H \otimes \mathbb{C}[\boldsymbol{Y}^{\pm 1}]$ *for* H *the finite Hecke algebra.*

Notice we can use the relation $T_i X_i T_i = X_{i+1}$ recursively to write any X_j in terms of $T_1, ..., T_{j-1}$ and X_1. But the penultimate relation tells us that $X_1 = qT_0T_0^\vee$. On the other hand, we can write X_j in terms of $T_j, ..., T_n$ and X_n. But the final relation says $X_n = (T_n^\vee T_n)^{-1}$. Therefore, the $\boldsymbol{X}$-generators can be replaced and the whole DAHA is generated by $T_0^\vee, T_0, T_1, ..., T_n, T_n^\vee$. This is discussed in [Sto05, Theorem 3.4] and is captured by the affine Coxeter diagram of type $\widetilde{C}_n$:



Slide 5 – Spherical Subalgebra

Our goal is to relate the DAHA to a commutative algebra, so that we can interpret it somewhat more geometrically. However, it is generically highly non-commutative (and its centre is trivial). That said, upon specialising $q = 1$, the DAHA turns out to have a large centre.

Remark The Hecke symmetriser $\mathbf{e} \in H$ and satisfies $T_i\mathbf{e} = \mathbf{e}T_i = \tau_i\mathbf{e}$ for all $i = 1, ..., n$.

The Hecke symmetriser kills (a lot of) the finite Hecke algebra in the above decomposition to make the algebra "more commutative". The subalgebras $\mathbf{e}\mathcal{H}_{n,\mathbf{t},q}\mathbf{e}$ are thought of as quantisation deformations of the spherical subalgebra $\mathbf{e}\mathcal{H}_{n,\mathbf{t},1}\mathbf{e}$ at the classical level. But why do we care?

Theorem *When $q = 1$, we have an isomorphism $Z(\mathcal{H}_{n,\mathbf{t},1}) \cong \mathbf{e}\mathcal{H}_{n,\mathbf{t},1}\mathbf{e}$ given by $z \mapsto z\mathbf{e}$.*

We now give a brief overview of some known results relating the DAHA to certain varieties.

Rational Cherednik Algebra

In [EG01], Pavel Etingof and Victor Ginzburg describe the centre $Z(\mathsf{H}_{0,c})$ of a so-called *rational limit* of the DAHA. They identify it with the quiver variety of the tadpole quiver, drawn below. We won't dwell much on this; something similar is done for the usual DAHA of type A_{n-1}.

DAHA of Type A_{n-1}

In [Obl03a], Alexei Oblomkov describes the centre $Z(H_{1,t})$ of the DAHA of type A_{n-1} by finding an isomorphism to the so-called Calogero-Moser space (which is really a type of *character variety*).

Definition The Calogero-Moser space CM_t is the (GIT-)quotient of the subset

$$\{(X, Y, u, v) : X^{-1}Y^{-1}XY - t^{-2}\mathbb{1}_n = u \otimes v\} \subseteq \mathrm{GL}_n \times \mathrm{GL}_n \times \mathbb{C}^n \times (\mathbb{C}^n)^\star$$

where the action by elements of $g \in \mathrm{GL}_n$ is given as follows:

$$g \bullet (X, Y, u, v) = (gXg^{-1}, gYg^{-1}, gu, vg^{-1}).$$

Proposition *The action of GL_n on CM_t is free.*

Sketch of Proof: The crux of the argument is to assume we have a non-trivial stabiliser, and show that this allows us to find a non-zero X- and Y-invariant *proper* subspace on which we can restrict these matrices. But no such subspace exists; see the proof of [Obl03a, Lemma 2.1]. $\square$

We will determine the eigenvalues of $(X, Y) \coloneqq X^{-1}Y^{-1}XY$ by acting on an arbitrary $w \in \mathbb{C}^n$.

- If $v(w) = 0$, then $(X, Y)w = t^{-2}w + 0$, giving an eigenvalue t^{-2} with multiplicity $n - 1$.
- But $\det\big(X^{-1}Y^{-1}XY\big) = 1$, which gives a remaining eigenvalue t^{2n-2} with multiplicity 1.

Generalised DAHA

Lastly, [EGO06] conjecture a similar result for some *generalised* DAHA; these are defined in terms of some *star-shaped* quiver Q. Specifically, their conjecture is about $Q = \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. Fortunately, there is a straightforward isomorphism between the generalised DAHA associated with $\widetilde{D}_4$ and the DAHA of type $C^\vee C_n$. Consequently, Problem 7 is refined in the sense that [EGO06] conjecture *specifically* the variety V we should be looking at.

Slide 6 – Character Varieties

We will interpret Definition 8 geometrically. First, let Σ_g be a compact Riemann surface with genus g. Then, the fundamental group $\pi_1(\Sigma_g)$ is generated by based loops $a_1, ..., a_g$ around each hole and $b_1, ..., b_g$ through each hole. The character variety here is $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{GL}_m)/\sim$, where GL_m acts on the group of homomorphisms by conjugation, and $\sim$ declares that two homomorphisms equivalent if the closures of their orbits intersect. Recall we have a presentation

$$\pi_1(\Sigma_g) = \langle a_1, ..., a_g, b_1, ..., b_g : (a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1}) = 1 \rangle.$$

If we remove k points, we obtain the compact Riemann surface with genus g and k *punctures*, denoted $\Sigma_{g,k}$. We have additional $\pi_1(\Sigma_{g,k})$-generators c_i around each puncture. Hence, we obtain

$$\pi_1(\Sigma_{g,k}) = \langle a_1, ..., a_g, b_1, ..., b_g, c_1, ..., c_k : (a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1}) c_1 \cdots c_k = 1 \rangle.$$

Remark The meaning of $/\!\!/$ is that we take only the closed orbits under the GL_m-conjugation action; this is because there may be singular points. However, we can impose certain conditions on the conjugacy classes to guarantee that the action is free; it is really the naïve quotient.

It is clear that $\widehat{\mathcal{M}}$, and thus the variety $\mathcal{M}_{g,k}$ itself, is empty if the product of the determinants

$$\prod_{i=1}^{k} \det(C_i) \neq 1.$$

Lemma ([HLRV11, Lemma 2.1.2]) *There exists a generic tuple of semi-simple conjugacy classes.*

The Calogero-Moser space CM_t in [Obl03a] is a character variety of the one-punctured torus. Indeed, the defining equation can be re-written as $(XYX^{-1}Y^{-1})A = \mathbb{1}_n$ and thus we see that $g = 1$, $k = 1$ and $m = n$. The genericity of the eigendata is not too bad to verify; it is almost immediate from looking at the eigenvalues and multiplicities we obtained earlier.

Slide 7 – Generic Semi-Simple Eigendata

We will give more of a flavour for how this genericity condition arises. Indeed, if a point in $\widehat{\mathcal{M}}$ has non-trivial stabiliser, then we can restrict to a subspace invariant under the matrices A_j, B_j, C_i (i.e. a non-simple subrepresentation of the fundamental group π_1). Assume $\prod_i \det(C_i) = 1$, so the character variety *might* be non-empty. The genericity condition in Definition 9 is equivalent to this: let $W \subseteq \mathbb{C}^m$ be a subspace stable under each matrix C_i. Then, the only two such subspaces are $W = 0$ or $W = \mathbb{C}^m$ that make the following product true: $\prod_i \det(C_i|_W) = 1$.

Slide 8 – Calogero-Moser Space in Type $C^\vee C_n$

The definition of Calogero-Moser space in our situation comes by fixing some analytic data $\boldsymbol{\lambda}$ and combinatorial data $\boldsymbol{\mu}$ in the way as presented; these define some diagonal conjugacy classes. Showing genericity requires a bit (but not too much) effort; it relies on the 'splitting' in $\mathcal{C}_4$.

Note the Calogero-Moser space $\mathsf{CM} = \mathcal{M}_{0,4}$ is a character variety of $\mathbb{CP}^1 \setminus \{4 \text{ points}\}$. We can compute its dimension by using the combinatorial formula from the previous slide. Indeed then,

$$\dim(\mathsf{CM}) = (0-2+4)(2n)^2 - 7n^2 - (n-1)^2 - 1 + 2 = 8n^2 - 7n^2 - n^2 + 2n - 1 - 1 + 2 = 2n.$$

Remark The Calogero-Moser space is so-named because it is the phase space of the corresponding *Calogero-Moser system*. There is interesting dynamics here, something we want to investigate soon. Based on [Cha19], our case should correspond to the Koornwinder-van Diejen system.

Slide 9 – Headline of the Talk

Alternatively, we can think of it as an isomorphism $\operatorname{Spec}(Z) \cong \mathsf{CM}$. We now want to associate to one-dimensional representations (characters $\chi : Z \to \mathbb{C}$) a point in Calogero-Moser space. We can use [Cha19] to interpret said restrictions as $2n \times 2n$ matrices. Such an association is also described in [EGO06]; see the next slide for a (very vague and) brief idea.

Slide 10 – A Map from the DAHA to Calogero-Moser Space

A proper set-up would sadly take too much time here. However, we aim to provide a flavour as to what is happening here. The idea comes from [EGO06, §5], who introduce a map that associates to a (certain class of) irreducible representation some point in the Calogero-Moser space.

Remark The idea is to analyse Φ in greater detail, and use theory from algebraic geometry to show it is an isomorphism; this is analogous to [EG01] and [Obl03a] in their respective settings.

The trick is to embed $Z \subseteq \mathcal{H}$ into a "bigger algebra" $\mathscr{D}_1$ generated by meromorphic functions $\mathbb{C}(\boldsymbol{X})$ and translations $\tau(\mathbb{Z}^n)$. If we localise the centre on some ideal generated by $\delta(\boldsymbol{X})$, then this is isomorphic to the W-invariant localised Laurent polynomials $\mathbb{C}[\boldsymbol{P}^{\pm 1}, \boldsymbol{X}^{\pm 1}]^W_{\delta(\boldsymbol{X})}$.

Remark In general, this is the algebra of q-difference operators and is denoted $\mathscr{D}_q$. But at the classical level $q = 1$, the lattice of translations $\tau(\mathbb{Z}^n)$ reduces to Laurent polynomials $\mathbb{C}[\boldsymbol{P}^{\pm 1}]$. To relate this to the affine/finite Weyl groups, we can encode s_0 by P_1. In this way, we obtain

$$\mathbb{C}(\boldsymbol{X}) \rtimes \widetilde{W} \cong \mathscr{D}_q \rtimes W.$$

Slide 11 – Multiplicative Quiver Varieties

Explicitly, a representation of the multiplicative preprojective algebra is a collection of vector spaces and linear maps $\{(X_v, f_a)\}_{v\in Q_0, a\in \overline{Q}_1}$ which form a representation of $\overline{Q}$ but satisfy these:

(i) For each $a \in \overline{Q}_1$, we have $\mathrm{id}_{X_{h(a)}} + (f_{a^*} \circ f_a)$ is invertible.

(ii) For each $v \in Q_0$, we have $\displaystyle\prod_{\substack{a\in\overline{Q}_1 \\ h(a)=v}} \Big(\mathrm{id}_{X_{h(a)}} + (f_{a^*}\circ f_a)\Big)^{\varepsilon(a)} = q_v \,\mathrm{id}_{X_v}$.

Remark We can view the space of representations as a level set of a *moment map*. Indeed, if we define $\Psi := \prod_a (1 + a^* a)^{\varepsilon(a)}$ and $q := \sum_v q_v e_v$, then $\mathrm{Rep}(\Lambda^{\mathbf{q}}, \boldsymbol{\alpha})$ is viewed as a level set $\{\Psi = q\}$, and is thus a closed sub-variety of the variety of representations of the double quiver.

Slide 12 – From Character to Quiver Varieties

In the case that the genus $g > 0$, one can work with a similar underlying quiver except there are now also g-many loops at the central vertex. This is discussed a bit more in [HLRV11].

Lemma ([CBS04, Lemma 1.5]) *If* $\prod_{v\in Q_0} q_v^{\alpha_v} \neq 1$*, the representation space* $\mathrm{Rep}(\Lambda^{\mathbf{q}}, \boldsymbol{\alpha})$ *is empty.*

On the level of quiver varieties then, we can think of this product condition as capturing the genericity we impose on the level of character varieties. This is made clear in the next example.

Example For the quiver variety associated with CM, we work with the following quiver data:

$$q_0 = -k_0^{-1}u_0^{-1}u_n^{-1}k_n, \qquad q_{[1,1]} = -k_0^2, \qquad q_{[2,1]} = -u_0^2,$$
$$q_{[3,1]} = -u_n^2, \qquad q_{[4,1]} = -k_n^{-2}t^2, \qquad q_{[4,2]} = t^{-2n}.$$

The dimension vector here is $\boldsymbol{\alpha} = (2n, n, n, n, n, 1)$ and we can quickly see that $\prod_{v\in Q_0} q_v^{\alpha_v} = 1$.

Slide 13 – Local Coordinates on Calogero-Moser Space

The defining equation on the Calogero-Moser space we work with is $A_1A_2A_3A_4 = \mathbb{1}_{2n}$. If we instead restrict to a three-tuple by defining $X := A_1A_2$, then we obtain two related problems:

$$A_1A_2X^{-1} = \mathbb{1}_{2n} \qquad \text{and} \qquad XA_3A_4 = \mathbb{1}_{2n}.$$

Example Suppose that $n = 1$ and we want to solve $A_1A_2A_3A_4 = \mathbb{1}_2$ (these are all 2×2 matrices). This is a well-studied problem with a relationship to Fuchsian systems and Painlevé theory (in particular to the *Painlevé VI equation*, some second-order ODE). We can use the above idea to obtain coordinates on the character variety. Begin by assuming that X is diagonalisable:

- Since $\det(X) = 1$, we know the eigenvalues are X_1 and X_1^{-1}; this gives us **coordinate one**.
- But the other problem also involves X. Since conjugation on the character variety is simultaneous (affecting all A_i concurrently), this lack of independence between the two equations gives rise to conjugation by $\mathrm{diag}(P_1, P_1^{-1})$; this gives us **coordinate two**.

We know that $\dim(\mathsf{CM}) = 2n = 2$ in this case, and we have two coordinates on this variety.

Not only did [Obl03b] discuss this $n = 1$ case in a separate paper, but he shows that the character variety is an explicit affine cubic. Moreover, he even considers Poisson structures on both sides.

Remark Geometrically, we have cut the four-punctured Riemann sphere $\Sigma_{0,4}$ into hemispheres, each containing two punctures. If we then contract each new boundary component to a point, we see that the two problems above correspond to the *three*-punctured Riemann sphere $\Sigma_{0,3}$.

The idea for higher-rank is similar: the first set of coordinates encodes the diagonal matrix X, and the second encodes the glueing of the two solutions for $\Sigma_{0,3}$ to obtain a solution for $\Sigma_{0,4}$. The aim now is to play these smaller problems off of each other and use [CBS04]:

- Analysing Q_2 tells us that X has distinct eigenvalues and the moduli space is *rigid*.
- Ensuring that the eigenvalues are *generic-enough*, they are pairwise reciprocal.
- This allows us to write X as a direct sum of 2×2 matrices.
- Such a solution also fixes A_3 and A_4 by the rigidity.

Remark By *generic-enough* we mean to localise by some function $\delta(\boldsymbol{X})$ of the eigenvalues. Think of the isomorphism as defining local coordinates on Calogero-Moser space. Obtaining explicit formulae for the A_i is difficult but possible by using the DAHA (i.e. the *Basic Representation*).

Slide 14 – The Duality Isomorphism

We apply the duality isomorphism to the elements A_1, A_2, A_3, A_4 to see what their corresponding elements are. It is a straightforward calculation using Proposition 18 to determine this:

$$A_1 \mapsto q^{-1}A_3^{-1}, \qquad A_2 \mapsto A_2^{-1}, \qquad A_3 \mapsto qA_1^{-1}, \qquad A_4 \mapsto A_4^{-1}.$$

In particular, we see that $X = A_1A_2 \mapsto (qA_2A_3)^{-1} = q^{-1}A_4A_1$ under the duality isomorphism. This acts as motivation for us defining the matrix $Y := A_2A_3$. Indeed, we now get a completely similar pair of problems on $\Sigma_{0,3}$ to that which we had when we set $X = A_1 A_2$. Namely, we have

$$A_2 A_3 Y^{-1} = \mathbb{1}_{2n} \qquad \text{and} \qquad A_1 Y A_4 = \mathbb{1}_{2n}.$$

The argument is identical to before: we get an isomorphism to an open subset of *this* Calogero-Moser space. But then we can pull this back to the usual Calogero-Moser space CM to obtain a second set of local coordinates. The idea is that ε on DAHAs induces a map $\varepsilon_{\mathsf{CM}}$ on Calogero-Moser spaces: for Φ the map in [EGO06, Proposition 5.2.10] which sends an irreducible module induced by a character to a point in Calogero-Moser space, we can say

$$\varepsilon_{\mathsf{CM}} \circ \Phi = \Phi \circ \varepsilon.$$

Slide 15 – Sketching the Main Argument

- *Nothing much else to say.*
- *Nothing much else to say.*
- We are really working with $\Phi = \varepsilon_{\mathsf{CM}}^{-1} \circ \Phi \circ \varepsilon$, where we defined $\varepsilon_{\mathsf{CM}}$ just above.
- The Cohen-Macaulay property allows us to extend to the whole spectrum, and irreducibility tells us that the extension is *dominant* (meaning its image is dense). One can then use a general theorem [Sha13, Theorem 2.21] about normal varieties to give the existence of a birational inverse and thus we are done.

Remark As mentioned, this is also conjectured for the GDAHA by [EGO06] in the case that the underlying quiver is instead $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. However, our method will not carry across because there is no obvious way to obtain coordinates on the character variety. Also, the DAHA is not well understood (there is no explicit Basic Representation and this is actually a key ingredient).