Double Affine Hecke Algebras and Calogero-Moser Space

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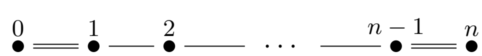
Ivan Cherednik [Che92] introduced the double affine Hecke algebra (DAHA) associated to a reduced irreducible root system in order to prove conjectures regarding Macdonald polynomials. Subsequent work by Masatoshi Noumi [Nou95] and Siddhartha Sahi [Sah99] extended the theory to cover the setting of an arbitrary irreducible *affine* root system. In particular, the non-reduced irreducible affine root system of type $C^\vee C_n$ contains, as subsystems, all non-reduced irreducible affine root systems of classical type; this is noted in Ian Macdonald's book [Mac03, p.12].

Slide 2 – Hecke Algebras of Type C_n

Recall a *reflection* in a root system R across the hyperplane perpendicular to $\alpha \in R$ is given by $s_\alpha(\beta) = \beta - 2\alpha(\alpha,\beta)/(\alpha,\alpha) =: \beta - (\alpha,\beta)\alpha^\vee$. The *simple reflections* are precisely those $s_i \coloneqq s_{a_i}$ across hyperplanes perpendicular to the simple roots of R. These are used in the next definition.

Definition The Weyl group of type C_n is generated by $s_1, ..., s_n$ which satisfy the quadratic relation $s_i^2 = 1$ for all $i = 1, ..., n$ and satisfy the braid relations $s_i s_j s_i \cdots = s_j s_i s_j \cdots$ where there are $\mathrm{ord}(s_i s_j)$-many terms on each side. The affine Weyl group of type $\widetilde{C}_n$ is generated by an additional element s_0 which satisfies the quadratic relation, as well as $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

The affine Coxeter diagram of type $\widetilde{C}_n$ encodes the braid relations in the affine Weyl group: $\mathrm{ord}(s_i s_j) = 2, 3, 4$ says the i^{th} and j^{th} nodes are connected by zero/one/two edges respectively.



Remark We can think of the quadratic relations as being deformed into the Hecke relations:

$$\underset{\text{inside } \widetilde{W}}{s^2 = 1} \quad \Rightarrow \quad \underset{\text{inside } \mathbb{C}\widetilde{W}}{(s_i - 1)(s_i + 1) = 0} \quad \rightsquigarrow \quad \underset{\text{inside } \widetilde{H}}{(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0.}$$

Slide 3 – Generators of the Affine Hecke Algebra

Approach 1: an affine Hecke algebra associated to the affine Weyl group $\widetilde{W}$ has basis $\{T_w\}_{w\in\widetilde{W}}$. The idea is to write any affine Weyl element $w = s_{i_1}\cdots s_{i_k}$ as a product of simple reflections in a *reduced* (minimal) way. Having defined $T_i := T_{s_i}$, this can be extended to $T_w := T_{i_1}\cdots T_{i_k}$.

Approach 2: the affine Weyl group $\widetilde{W} \cong W \ltimes \tau(\mathbb{Z}^n)$, where $\tau(\mathbb{Z}^n)$ are translations in an integer lattice (really the coroot lattice $Q^\vee$ associated with the root system of type C_n). A *translation* in the direction of $\lambda \in Q^\vee$ is given by $t_\lambda(x) = x + \lambda$. The Y_i are now Hecke elements corresponding to a translation in the direction of ε_i.

Slide 4 – The DAHA of Type $C^\vee C_n$

We can clearly see the finite Hecke algebra of type $C^\vee C_n$ inside Definition 5 with the quoted braid relations, and an obvious copy of one affine Hecke algebra of type $\widetilde{C}_n$ sitting inside. However, the second affine Hecke algebra is a little more obscure. We now justify why the DAHA contains two copies of an affine Hecke algebra that overlap on a finite Hecke algebra.

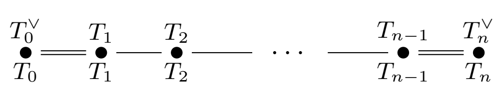
We can use the theorem of Lusztig to obtain a linear basis of the DAHA. Indeed, the previous result tells us that X_i pairwise commute and generate a subalgebra $\mathbb{C}[\boldsymbol{X}^{\pm 1}]$. Hence, since the DAHA is the affine Hecke algebra with this extra collection of elements, we have an isomorphism $\mathcal{H} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}] \otimes \widetilde{H}$ as vector spaces, where $\widetilde{H}$ is the affine Hecke algebra.

Corollary *As vector spaces, we have* $\mathcal{H} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}] \otimes H \otimes \mathbb{C}[\boldsymbol{Y}^{\pm 1}]$ *for* H *the finite Hecke algebra.*

Notice we can use the relation $T_i X_i T_i = X_{i+1}$ recursively to write any X_j in terms of $T_1, ..., T_{j-1}$ and X_1. But the penultimate relation tells us that $X_1 = qT_0T_0^\vee$. On the other hand, we can write X_j in terms of $T_j, ..., T_n$ and X_n. But the final relation says $X_n = (T_n^\vee T_n)^{-1}$. Thus, the $\boldsymbol{X}$-generators can be replaced and the whole DAHA is generated by $T_0^\vee, T_0, T_1, ..., T_n, T_n^\vee$. This is made clear in [Sto05, Theorem 3.4]. Explicitly, each X_i can be written in the following ways:

$$X_i = qT_{i-1}\cdots T_1 T_0 T_0^\vee T_1 \cdots T_{i-1} \qquad \text{and} \qquad X_i = T_i^{-1}\cdots T_n^{-1}(T_n^\vee)^{-1}T_{n-1}^{-1}\cdots T_i^{-1}.$$

A neat way to visualise this is via the following labelled affine Coxeter diagram of type $\widetilde{C}_n$:



Remark Reading the above generators clockwise gives us the relation $qT_0T_0^\vee ST_n^\vee T_n S^\dagger = 1$, where $S := T_1 \cdots T_{n-1}$ and $S^\dagger := T_{n-1}\cdots T_1$. Moreover, the top/bottom rows satisfy respective affine Hecke relations, along with the compatibility commutation conditions $[T_0, T_n^\vee] = [T_0^\vee, T_n] = 0$.

Slide 5 – Spherical Subalgebra

Our goal is to relate the DAHA to a commutative algebra, so that we can interpret it somewhat more geometrically. However, it is generically highly non-commutative (and its centre is trivial). That said, upon specialising $q = 1$, the DAHA turns out to have a large centre.

Remark The Hecke symmetriser $\mathbf{e} \in H$ and satisfies $T_i\mathbf{e} = \mathbf{e}T_i = \tau_i\mathbf{e}$ for all $i = 1, ..., n$. One can view the subalgebras $\mathbf{e}\mathcal{H}_{n,\mathbf{t},q}\mathbf{e}$ as quantisation deformations of $\mathbf{e}\mathcal{H}_{n,\mathbf{t},1}\mathbf{e}$ (the "classical level").

Rational Cherednik Algebra

In [EG01], Pavel Etingof and Victor Ginzburg describe the centre $Z(\mathsf{H}_{0,c})$ of a so-called *rational limit* of the DAHA. They identify it with the quiver variety of the tadpole quiver, drawn below. We won't dwell much on this; something similar is done for the usual DAHA of type A_{n-1}.

DAHA of Type A_{n-1}

In [Obl03a], Alexei Oblomkov describes the centre $Z(H_{1,t})$ of the DAHA of type A_{n-1} by finding an isomorphism to the so-called Calogero-Moser space (an example of a *character variety*).

Definition The Calogero-Moser space CM_t is the (GIT-)quotient of the subset

$$\{(X, Y, u, v) : XYX^{-1}Y^{-1} - t^{-2}\mathbb{1}_n = u \otimes v\} \subseteq \mathrm{GL}_n \times \mathrm{GL}_n \times \mathbb{C}^n \times (\mathbb{C}^n)^\star$$

where the action by elements of $g \in \mathrm{GL}_n$ is given as follows:

$$g \bullet (X, Y, u, v) = (gXg^{-1}, gYg^{-1}, gu, vg^{-1}).$$

Proposition *The action of* GL_n *on* CM_t *is free.*

We will determine the eigenvalues of $(X, Y) := XYX^{-1}Y^{-1}$ by acting on an arbitrary $w \in \mathbb{C}^n$.

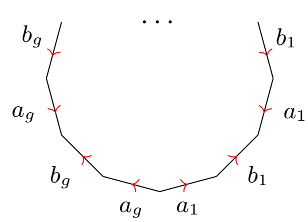
- If $v(w) = 0$, then $(X, Y)w = t^{-2}w + 0$, giving an eigenvalue t^{-2} with multiplicity $n - 1$.
- But $\det\big(X^{-1}Y^{-1}XY\big) = 1$, which gives a remaining eigenvalue t^{2n-2} with multiplicity 1.

Generalised DAHA

Lastly, [EGO06] conjecture a similar result for some *generalised* DAHA; these are defined in terms of some *star-shaped* quiver Q. Specifically, their conjecture is about $Q = \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. Fortunately, there is a straightforward isomorphism between the generalised DAHA associated with $\widetilde{D}_4$ and the DAHA of type $C^\vee C_n$. Consequently, Problem 8 is refined in the sense that [EGO06] conjecture *specifically* the variety V we should be looking at.

Slide 6 – Character Varieties

We will interpret Definition 9 more geometrically. First, let Σ_g be a compact Riemann surface with genus g. Then, the fundamental group $\pi_1(\Sigma_g)$ is generated by based loops $a_1, ..., a_g$ around each hole and $b_1, ..., b_g$ through each hole. The classic picture representing this is the following.



The character variety here is $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{GL}_m)/\sim$, where GL_m acts on the group of homomorphisms by conjugation, and $\sim$ declares that two homomorphisms equivalent if the closures of their orbits intersect. Recall we have a presentation of the fundamental group of Σ_g, namely

$$\pi_1(\Sigma_g) = \langle a_1, ..., a_g, b_1, ..., b_g : (a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1}) = 1 \rangle.$$

If we remove k points, we obtain the compact Riemann surface with genus g and k *punctures*, denoted $\Sigma_{g,k}$. We have additional $\pi_1(\Sigma_{g,k})$-generators c_i around each puncture. Hence, we obtain

$$\pi_1(\Sigma_{g,k}) = \langle a_1, ..., a_g, b_1, ..., b_g, c_1, ..., c_k : (a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_g b_g a_g^{-1} b_g^{-1}) c_1 \cdots c_k = 1 \rangle.$$

Remark The meaning of $/\!\!/$ is that we take only the closed orbits under the GL_m-conjugation action; this is because there may be singular points. However, we can impose certain conditions on the conjugacy classes to guarantee that the action is free; it is really the naïve quotient.

It is clear that $\widehat{\mathcal{M}}$, and thus the variety $\mathcal{M}_{g,k}$ itself, is empty if the product of the determinants

$$\prod_{i=1}^{k} \det(C_i) \neq 1.$$

Lemma ([HLRV11, Lemma 2.1.2]) *There exists a generic tuple of semi-simple conjugacy classes.*

The Calogero-Moser space CM_t in [Obl03a] is a character variety of the one-punctured torus: the defining equation can be written as $(XYX^{-1}Y^{-1})A = \mathbb{1}_n$ and thus $g = 1$, $k = 1$ and $m = n$.

Remark In fact, [HLRV11] conjecture a relationship with Hodge numbers (open) but still proved connectedness in a sequel paper [HLRV13]. A short while later, Mellit in [Mel20] gives the proof of a formula for the Poincaré polynomial of $\mathcal{M}_{g,k}$ conjectured in the first paper referenced here.

Slide 7 – Generic Semi-Simple Eigendata

We will give more of a flavour for how this genericity condition arises. Indeed, if a point in $\widehat{\mathcal{M}}$ has non-trivial stabiliser, then we can restrict to a subspace invariant under the matrices A_j, B_j, C_i (i.e. a non-simple subrepresentation of the fundamental group π_1). Assume $\prod_i \det(C_i) = 1$, so the character variety *may* be non-empty. The genericity condition in Definition 10 is equivalent to this: let $W \subseteq \mathbb{C}^m$ be a subspace stable under the action of each matrix C_i. Then, the only two such subspaces that ensure $\prod_i \det\big(C_i\big|_W\big) = 1$ are the following: $W = \{0\}$ or $W = \mathbb{C}^m$.

Slide 8 – Calogero-Moser Space in Type $C^\vee C_n$

The definition of Calogero-Moser space in our situation comes by fixing some analytic data $\boldsymbol{\lambda}$ and combinatorial data $\boldsymbol{\mu}$ in the way as presented; these define some diagonal conjugacy classes. Showing genericity requires a bit (but not too much) effort; it relies on the 'splitting' in $\mathcal{C}_4$.

The Calogero-Moser space $\mathsf{CM} = \mathcal{M}_{0,4}$ is a character variety of $\mathbb{CP}^1 \setminus \{4 \text{ points}\}$. We calculate

$$\dim(\mathsf{CM}) = (0-2+4)(2n)^2 - 7n^2 - (n-1)^2 - 1 + 2 = 8n^2 - 7n^2 - n^2 + 2n - 1 - 1 + 2 = 2n.$$

Remark The Calogero-Moser space is so-named because it is the phase space of the corresponding *Calogero-Moser system*. There is interesting dynamics here, something we want to investigate soon. Based on [Cha19], our case should correspond to the Koornwinder-van Diejen system.

Slide 9 – Headline of the Talk

Alternatively, one can think of it as an isomorphism $\mathrm{Spec}(Z) \cong \mathsf{CM}$. We can pass from R to $\mathrm{Spec}(R)$ since, for a commutative ring R and $x = \mathfrak{p} \in \mathrm{Spec}(R)$, we can associate to any $r \in R$ a function on the spectrum by defining $r(x)$ to be the image of r under the canonical maps

$$R \twoheadrightarrow R/\mathfrak{p} \hookrightarrow \mathrm{Quot}(R/\mathfrak{p}).$$

Slide 10 – A Map from the DAHA to Calogero-Moser Space

A proper set-up would sadly take too much time here. However, we aim to provide a flavour as to what is happening here. The idea comes from [EGO06, §5], who introduce a map that associates to a (certain class of) irreducible representation some point in the Calogero-Moser space.

Remark The idea is to analyse Φ in greater detail, and use theory from algebraic geometry to show it is an isomorphism; this is analogous to [EG01] and [Obl03a] in their respective settings.

Slide 11 – Multiplicative Quiver Varieties

Explicitly, a representation of the multiplicative preprojective algebra is a collection of vector spaces and linear maps $\{(X_v, f_a)\}_{v\in Q_0, a\in\overline{Q}_1}$ which form a representation of $\overline{Q}$ but satisfy these:

(i) For each $a \in \overline{Q}_1$, we have $\mathrm{id}_{X_{h(a)}} + (f_{a^*} \circ f_a)$ is invertible.

(ii) For each $v \in Q_0$, we have $\displaystyle\prod_{\substack{a\in\overline{Q}_1 \\ h(a)=v}} \Big(\mathrm{id}_{X_{h(a)}} + (f_{a^*} \circ f_a)\Big)^{\varepsilon(a)} = q_v \,\mathrm{id}_{X_v}$.

Slide 12 – From Character to Quiver Varieties

In the case that the genus $g > 0$, one can work with a similar underlying quiver except there are now also g-many loops at the central vertex. This is discussed a bit more in [HLRV11].

Lemma ([CBS04, Lemma 1.5]) *If* $\prod_{v\in Q_0} q_v^{\alpha_v} \neq 1$*, the representation space* $\mathrm{Rep}(\Lambda^{\mathbf{q}}, \boldsymbol{\alpha})$ *is empty.*

On the level of quiver varieties then, we can think of this product condition as capturing the genericity we impose on the level of character varieties. This is made clear in the next example.

Example For the quiver variety associated with CM, we work with the following quiver data:

$$q_0 = -k_0^{-1}u_0^{-1}u_n^{-1}k_n, \qquad q_{[1,1]} = -k_0^2, \qquad q_{[2,1]} = -u_0^2,$$
$$q_{[3,1]} = -u_n^2, \qquad q_{[4,1]} = -k_n^{-2}t^2, \qquad q_{[4,2]} = t^{-2n}.$$

The dimension vector here is $\boldsymbol{\alpha} = (2n, n, n, n, n, 1)$ and we can quickly see that $\prod_{v\in Q_0} q_v^{\alpha_v} = 1$.

Slide 13 – Local Coordinates on Calogero-Moser Space

The defining equation on the Calogero-Moser space we work with is $A_1A_2A_3A_4 = \mathbb{1}_{2n}$. If we instead restrict to a three-tuple by defining $X := A_1A_2$, then we obtain two related problems:

$$A_1A_2X^{-1} = \mathbb{1}_{2n} \qquad \text{and} \qquad XA_3A_4 = \mathbb{1}_{2n}.$$

Example Suppose that $n = 1$ and we want to solve $A_1A_2A_3A_4 = \mathbb{1}_2$ (these are all 2×2 matrices). This is a well-studied problem with a relationship to Fuchsian systems and Painlevé theory (in particular to the *Painlevé VI equation*, some second-order ODE). We can use the above idea to obtain coordinates on the character variety. Begin by assuming that X is diagonalisable:

- Since $\det(X) = 1$, we know the eigenvalues are X_1 and X_1^{-1}; this gives us **coordinate one**.

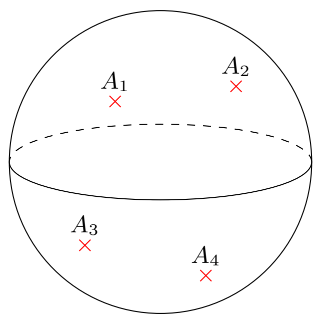
- But the other problem also involves X. Since conjugation on the character variety is simultaneous (affecting all A_i concurrently), this lack of independence between the two equations gives rise to conjugation by $\text{diag}(P_1, P_1^{-1})$; this gives us **coordinate two**.

We know that $\dim(\mathsf{CM}) = 2n = 2$ in this case, and we have two coordinates on this variety.

Not only did [Obl03b] discuss this $n = 1$ case in a separate paper, but he shows that the character variety is an explicit affine cubic. Moreover, he even considers Poisson structures on both sides.

Remark Geometrically, we have cut the four-punctured Riemann sphere $\Sigma_{0,4}$ into hemispheres, each containing two punctures. If we then contract each new boundary component to a point, we see that the two problems above correspond to the *three*-punctured Riemann sphere $\Sigma_{0,3}$.

The idea for higher-rank is similar: the first set of coordinates encodes the diagonal matrix X, and the second encodes the glueing of the two solutions for $\Sigma_{0,3}$ to obtain a solution for $\Sigma_{0,4}$.



The aim now is to play these smaller problems off of each other and use [CBS04]:

- Analysing Q_2 tells us that X has distinct eigenvalues and the moduli space is *rigid*.
- Ensuring that the eigenvalues are *generic-enough*, they are pairwise reciprocal.
- This allows us to write X as a direct sum of 2×2 matrices.
- Such a solution also fixes A_3 and A_4 by the rigidity.

Remark By "generic-enough", we mean to localise by some function $\delta(\boldsymbol{X})$ of the eigenvalues.

Slides 14 to 17 – The Matrices

As noted in Theorem 18, the map Φ restricts to an isomorphism when we localise by an ideal generated by a function of $X_1, ..., X_n$. Obtaining the matrices explicitly is non-trivial but possible when working with the so-called *Basic Representation* of the DAHA. Note that A_1 and A_2 are in terms of rational functions of $\boldsymbol{X}$ and $\boldsymbol{P}$ variables, whereas A_3 and A_4 are in terms only of $\boldsymbol{X}$ variables.

Slide 18 – The Duality Isomorphism

Definition The induced map $\varepsilon_{\mathsf{CM}}$ is determined by Proposition 19 at the DAHA level, that is

$$A_1 \mapsto q^{-1}A_3^{-1}, \qquad A_2 \mapsto A_2^{-1}, \qquad A_3 \mapsto qA_1^{-1}, \qquad A_4 \mapsto A_4^{-1}.$$

In particular, we see that $X = A_1A_2 \mapsto (qA_2A_3)^{-1} = q^{-1}A_4A_1$ under the duality isomorphism. This acts as motivation for us defining the matrix $Y \coloneqq A_2A_3$. Indeed, we now get a completely similar pair of problems on $\Sigma_{0,3}$ to that which we had when we set $X = A_1A_2$. Namely, we have

$$A_2A_3Y^{-1} = \mathbb{1}_{2n} \qquad \text{and} \qquad A_1YA_4 = \mathbb{1}_{2n}.$$

The argument is identical to before: we get an isomorphism to an open subset of *this* Calogero-Moser space. But we can pull this back to the usual Calogero-Moser space CM to obtain a second set of local coordinates on the Calogero-Moser space associated to our (pre-duality) DAHA.

Remark For the map Φ as in [EGO06, Proposition 5.2.10], which sends an irreducible module induced by a character to a point in Calogero-Moser space, we can say that $\varepsilon_{\mathsf{CM}} \circ \Phi = \Phi \circ \varepsilon$.

Slide 19 – Sketching the Main Argument

- *Nothing much else to say.*
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- We are really working with $\Phi = \varepsilon_{\mathsf{CM}}^{-1} \circ \Phi \circ \varepsilon$, where we defined $\varepsilon_{\mathsf{CM}}$ just above.
- The Cohen-Macaulay property allows us to extend to the whole spectrum, and irreducibility tells us that the extension is *dominant* (meaning its image is dense). One can then use a general theorem [Sha13, Theorem 2.21] about normal varieties to give the existence of a birational inverse and thus we are done.

Remark As mentioned, this is also conjectured for the GDAHA by [EGO06] in the case that the underlying quiver is instead $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. However, our method will not carry across because there is no obvious way to obtain coordinates on the character variety. Also, the DAHA is not well understood (there is no explicit Basic Representation and this is actually a key ingredient).