Real Analysis

Prison Mathematics Project

Introduction

Hello and welcome to the module on Real Analysis! What follows is a module intended to support the reader in learning this fascinating topic. The Prison Mathematics Project (PMP) realises that you may be practising mathematics in an environment that is highly restrictive, so this text can both be used independently and does not require a calculator.

What is Real Analysis?

You may or may not be used to calculus (for familiarity, you can also find notes made through the PMP on this topic) but it doesn't matter so much. We will introduce calculus informally and then make everything precise; this is the job of analysis. If any of these words seem a bit foreign, don't fret! We will provide a self-contained introduction to everything required for the understanding of this topic. Because this is an introductory course, we will restrict ourselves to looking at real functions of one variable only; although we could go wild and consider multivariable functions, we will *broadly* stick to what we can imagine in our heads.

Learning in this Module

The best way to learn mathematics is to do mathematics. Indeed, education isn't something that happens more than it is something we should all participate in. You will find various exercise questions and worked examples in these notes so that you may try to solve problems and deepen your understanding of this topic. Although the aim is for everything to only require the content of this module, you are encouraged to use any other sources you have at your disposal.

Acknowledgements

These notes are based on lecture courses by J. M. Speight at the University of Leeds.

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1 Preliminaries

2 Sequences

Now we have the foundations set by Chapter **??**, we can formalise the notion of a limit of a sequence. We may have been introduced to the notation $\lim_{x \to a}$ before but this is not quite rigorous.

Definition 2.1 A real sequence is a function $a : \mathbb{Z}^+ \to \mathbb{R}$ which assigns to each positive integer n a real number $a(n)$, which we henceforth write a_n. The sequence is denoted (a_n).

Example 2.2 Here are some examples of sequences.

(i) The sequence (a_n) where $a_n = n$ is the identity sequence $(1, 2, 3, 4, ...)$.

(ii) The sequence (b_n) where $b_n = n^2$ is the sequence $(1, 4, 9, 16, ...)$.

(iii) The sequence (c_n) where $c_n = 1 + 5/n^2$ is the sequence $(6, 9/4, 14/9, 21/16, ...)$.

Exercise 1 Find the first n with $c_n = 1$ to five decimal places, for (c_n) in Example 2.2.

[**Hint:** This means that $0.999995 \leq c_n < 1.000005$, so simply rearrange this for n.]

Definition 2.3 An inductively-defined sequence is a sequence (a_n) where a_1 is given and a_{n+1} is stated in terms of a_n, i.e. each entry of (a_n) is defined by the previous entry.

Example 2.4 Consider the sequence (a_n) where $a_1 = 1$ which is inductively-defined as follows:

$$a_{n+1} = \frac{a_n}{1 + a_n^2}.$$

We can determine the first four values, say, by *iterating*, that is find the next value, substitute it in, find the next value, and so on. Clearly, the first value $a_1 = 1$ (because it is given). Now then,

$$\begin{aligned} a_2 &= \frac{a_1}{1 + a_1^2} = \frac{1}{1 + 1^2} = \frac{1}{2}, \\ a_3 &= \frac{a_2}{1 + a_2^2} = \frac{1/2}{1 + (1/2)^2} = \frac{2}{5}, \\ a_4 &= \frac{a_3}{1 + a_3^2} = \frac{2/5}{1 + (2/5)^2} = \frac{10}{29}. \end{aligned}$$

In fact, if we 'went to infinity' with this, we would see that the output would be 'very' close to one. We will actually prove later that this is true irregardless of what the starting value a_1 is (but note that this is **not** the case for **all** inductively-defined sequences).

To talk about convergence, that is the notion of a limit, when discussing real sequences, we need to use the absolute value. Recall that Definition **??** told us the absolute value of an integer, but we can define this concept for $\mathbb{R}$ in the exact same way, albeit more rigorously now we know what a function is.

Definition 2.5 The absolute value is a function $|\cdot| : \mathbb{R} \to \mathbb{R}$ defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0 \end{cases}.$$

Note: Recall the more succinct way to write the function in Definition 2.5: $|x| = \sqrt{x^2}$.

Exercise 2 Explain the properties $x \leq |x|$ and $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.

Proposition 2.6 (Triangle Inequality) *Let $x, y \in \mathbb{R}$. Then, $|x + y| \leq |x| + |y|$.*

Proof: Assume to the contrary that $|x + y| > |x| + |y|$. Because both sides are non-negative, we get the following chain of implications when we square the assumed inequality:

$$\begin{aligned} & & |x+y|^2 &> (\,|x| + |y|)^2 \\ &\Rightarrow & (x+y)^2 &> |x|^2 + 2|x||y| + |y|^2 \\ &\Rightarrow & x^2 + 2xy + y^2 &> |x|^2 + 2|xy| + |y|^2 \\ &\Rightarrow & xy &> |xy|, \end{aligned}$$

but this is a contradiction to Exercise 2. □

Definition 2.7 A real sequence (a_n) converges to a number $L \in \mathbb{R}$ if, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. This is denoted $a_n \to L$. The number L is called the limit of (a_n) and may be denoted either $\lim a_n$ or $\lim_{n\to\infty} a_n$.

Remark 2.8 Let's take a breather; Definition 2.7 is the first rigorous definition of a limit we have encountered, and other definitions will be built from it. Thus, it is important you have an idea of what this definition says. Indeed, we will explain it in words and we will provide a geometric interpretation (picture).

(i) Given a sequence (a_n), we can show that it 'approaches' the number L as $n \in \mathbb{Z}^+$ gets large by showing that for **any** positive number ($\varepsilon > 0$), there exists a point in the sequence a_N (there exists $N \in \mathbb{Z}^+$) after which (for all $n \geq N$) every term in the sequence lies within distance that positive number of the number L ($|a_n - L| < \varepsilon$). Because this needs to work for **any** ε, the idea is that the distance can be as large or as small as you like and we should still be able to find $N \in \mathbb{Z}^+$ to make this work.

(ii) Geometrically, this means that, if we plot n against a_n on a pair of axes, then after N, every pair of points will live inside a rectangle with width 2ε centred on the line $a_n = L$.

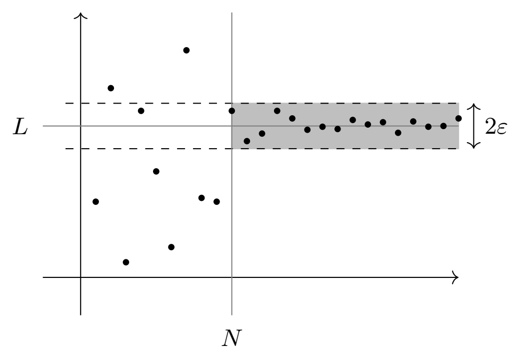

Figure 1: The geometric interpretation of the convergence of some sequence (a_n).

Definition 2.9 An ε-N proof is a proof of convergence using Definition 2.7. Although this terminology is used elsewhere, a proof from first principles also refers to an ε-N proof.

Note: When reading an ε-N proof, it may be that the writer has chosen a very specific (and/or complicated) expression for N in terms of ε; this is fine because it is expected that how far along our sequence we need to go before all the points are in the desired region, as in Figure 1, will very much depend on how wide our region is. However, when you are asked to **do** an ε-N proof, choosing this N isn't just by magic. You must 'estimate' (get a rigorous upper bound on) the quantity $|a_n - L|$, from which the choice should be clear.

Example 2.10 Suppose we wish to prove from first principles that, for $a_n = 1 + 5/n^2$, we have $a_n \to 1$. We first look at the quantity $|a_n - 1|$. Indeed,

$$\begin{aligned} |a_n - 1| &= \left|1 + 5/n^2 - 1\right| \\ &= \left|5/n^2\right| \\ &= 5/n^2 \\ &\leq 5/n. \end{aligned}$$

All this uses is the fact that the absolute value of a positive quantity, e.g. $5/n^2$, is the same as the positive quantity (this allowed us to ditch the absolute value on the third line) and that, for all $n \in \mathbb{Z}^+$, we have $n^2 \geq n$ which is equivalent to $1/n^2 \leq 1/n$ (this allowed us to simplify the denominator in the last line). Recall that we consider all $n \geq N$ when looking at the limit, which means that $1/n \leq 1/N$. In particular, we can add an additional inequality:

$$\leq 5/N.$$

Thus, to ensure that this is less than any given $\varepsilon > 0$, we just rearrange $5/N < \varepsilon$ to get that $N > 5/\varepsilon$. One final problem? How do we know that there exists an integer N satisfying this? By the Archimedean Property of $\mathbb{R}$, of course. Thus, we are set. Here is what the proof would look like (i.e. where we hide all of our thought processes and scrap work from view).

Proof: Let $\varepsilon > 0$ be given. There is $N \in \mathbb{Z}^+$ with $N > 5/\varepsilon$ by the Archimedean Property of $\mathbb{R}$. Then, for all $n \geq N$, it follows that

$$\begin{aligned} |a_n - 1| &= \left|1 + 5/n^2 - 1\right| \\ &= \left|5/n^2\right| \\ &= 5/n^2 \\ &\leq 5/n \\ &\leq 5/N \\ &< \varepsilon. \end{aligned}$$

Therefore, we can conclude that $a_n \to 1$. □

Exercise 3 Give a direct ε-N proof that the sequence (a_n) where $a_n = 1/n$ converges.

[**Hint:** You must first suggest what the limit L is and then go ahead with the proof.]

Exercise 4 Give a direct ε-N proof that the so-called *constant sequence* (b_n) converges, that is the sequence is defined by $b_n = k$ for all n, where $k \in \mathbb{R}$ is some fixed number.

Example 2.11 (Harder) We will give a direct ε-N proof that the following is true:

$$a_n = \frac{n^2 - \sin(n)}{(2n-7)(3n+1)} \to \frac{1}{6}.$$

We will **not** write this up formally, rather we perform the informal 'estimation' and discuss how we choose N to ensure that $\left|a_n - 1/6\right| < \varepsilon$. So, let's work with this absolute value, as usual:

$$\begin{aligned}
\left|a_n - 1/6\right| &= \left|\frac{n^2 - \sin(n)}{(2n-7)(3n+1)} - \frac{1}{6}\right| \\
&= \left|\frac{6n^2 - 6\sin(n) - (2n-7)(3n+1)}{6(2n-7)(3n+1)}\right| \\
&= \frac{1}{6}\left|\frac{19n - 6\sin(n) + 7}{(2n-7)(3n+1)}\right| \\
&= \frac{1}{6}\frac{\left|19n - 6\sin(n) + 7\right|}{(2n-7)(3n+1)}, \text{ if } n \geq 4, \\
&\leq \frac{1}{6}\frac{19n + 6\left|\sin(n)\right| + 7}{(2n-7)(3n+1)}, \text{ by the Triangle Inequality,} \\
&\leq \frac{1}{6}\frac{32n}{(2n-7)(3n+1)}, \text{ since } \left|\sin(n)\right| \leq 1 \leq n, \\
&\leq \frac{1}{6}\frac{32n}{(2n-n)(3n)}, \text{ if } n \geq 7, \\
&= \frac{2}{n}.
\end{aligned}$$

Therefore, to ensure that the final line $2/N < \varepsilon$, we need to choose $N > 2/\varepsilon$. However, we are not done yet; as you will have noticed, there were two extra stipulations we used to get to our 'nice' estimate: $n \geq 4$ and $n \geq 7$. Of course, we really only need to consider $n \geq 7$ because this automatically covers the $n \geq 4$ situation. Consequently, we just need to choose $N > \max\{2/\varepsilon, 7\}$ to make the direct proof work. Why? Because then, for all $n \geq N$, it is true that both $n \geq 7$ and that $2/n \leq 2/N < \varepsilon$.

Definition 2.12 A real sequence (a_n) diverges if it does **not** converge. If the sequence does not converge to a specific value $L \in \mathbb{R}$, then we denote this by $a_n \not\to L$.

Note: There is a subtlety: a divergent sequence (a_n) is one where $a_n \not\to L$ for **all** $L \in \mathbb{R}$.

Example 2.13 We will show directly, that is using an ε-N argument, that the sequence (a_n) given by $a_n = (-1)^n$ diverges. Assume to the contrary that $a_n \to L$ for some $L \in \mathbb{R}$. Then, this means that for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. Because this is true for **any** ε, it is certainly true for a specific choice, namely $\varepsilon = 1/2$. Now, note that $N, N+1 \in \mathbb{Z}^+$ are consecutive, so one is odd and the other is even. Thus, $|a_{N+1} - a_N| = 2$. But from the Triangle Inequality, we get

$$|a_{N+1} - a_N| = |a_{N+1} - L + L - a_N|$$

$$
\begin{aligned}
&\leq |a_{N+1} - L| + |L - a_N| \\
&= |a_{N+1} - L| + |a_N - L| \\
&< 1/2 + 1/2 \\
&= 1.
\end{aligned}
$$

Therefore, $2 = |a_{N+1} - a_N| < 1$, a clear contradiction. Thus, no such L exists so (a_n) diverges.

Exercise 5 Prove that the sequence (a_n) given by $a_n = n$ is divergent.

Thus far, we have used the language *the* limit, but this does require proof. Strictly speaking, we really should be saying that L is *a* limit of (a_n) if Definition 2.7 is satisfied. Thankfully, the proof is relatively simple.

Proposition 2.14 (Uniqueness of Limits) *The limit of a convergent sequence is unique.*

Proof: Suppose that (a_n) converges to both $L_1 \in \mathbb{R}$ and $L_2 \in \mathbb{R}$ and let $\varepsilon > 0$:

- $a_n \to L_1$ means there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, we have $|a_n - L_1| < \varepsilon/2$.
- $a_n \to L_2$ means there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$, we have $|a_n - L_2| < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$,

$$
\begin{aligned}
|L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \\
&\leq |a_n - L_1| + |a_n - L_2| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}
$$

But ε is arbitrary, so $|L_1 - L_2| < \varepsilon$ is equivalent to $|L_1 - L_2| = 0$, which is to say $L_1 = L_2$. □

Definition 2.15 Consider some real sequence (a_n).

(i) We say (a_n) is bounded above if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all n.

(ii) We say (a_n) is bounded below if there exists $K \in \mathbb{R}$ such that $a_n \geq K$ for all n.

Example 2.16 Here are some examples and non-examples of bounded sequences.

(i) The sequence defined by $a_n = n$ is bounded below but **not** above.

(ii) The sequence defined by $b_n = -n^5$ is bounded above but **not** below.

(iii) The sequence defined by $c_n = 3$ is bounded (above **and** below).

(iv) The sequence defined by $d_n = (-1)^n n$ is **not** bounded (above or below).

Proposition 2.17 *Any convergent sequence is bounded.*

Proof: Let (a_n) be a sequence such that $a_n \to L$. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. In particular, this holds for $\varepsilon = 1$, i.e. we can choose N to ensure that $|a_n - L| < 1$ for $n \geq N$. Now, consider the number

$$M = \max\{\, |a_1 - L|, \ldots, |a_N - L|, 1\},$$

which certainly exists because the maximum is taken from a finite set. It is now clear that $|a_n - L| \leq M$ for all n, which is equivalent to saying that $L - M \leq a_n \leq L + M$, by definition of the absolute value. Thus, the sequence is bounded above and below, i.e. (a_n) is bounded. □

Remark 2.18 In the proof of Proposition 2.17, where did M come from? Well, by the assumption that (a_n) converges, we know that for large enough n, that is for all $n \geq N$, we would have $|a_n - L| < 1$, so the 'tail' of the sequence is bounded below by $L - 1$ and above by $L + 1$. However, there is no guarantee that the first $N - 1$ terms will fall in this interval, so we must take the largest distance these points gets from the limit; this is where M comes from.

Exercise 6 State the converse to Proposition 2.17 and determine if it is true.

Theorem 2.19 (Algebra of Limits) *Let (a_n) and (b_n) be convergent sequences such that $a_n \to A$ and $b_n \to B$. Then, the following are true.*

(i) $a_n + b_n \to A + B$.

(ii) $a_n b_n \to AB$.

(iii) $1/a_n \to 1/A$ *so long as* $a_n \neq 0$ *for any* n *and* $A \neq 0$.

Proof: (i) See Exercise 7.

(ii) Let $\varepsilon > 0$. By Proposition 2.17, (b_n) is is bounded since it is assumed convergent. Indeed, there exists $K > 0$ such that $|b_n| < K$. Define the number $\varepsilon' := \varepsilon/(K + |A|) > 0$ and consider what it means for these sequences to converge:

- $a_n \to A$ means there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, we have $|a_n - A| < \varepsilon'$.

- $b_n \to B$ means there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$, we have $|b_n - B| < \varepsilon'$.

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$,

$$\begin{aligned}
|a_n b_n - AB| &= \big|(a_n - A)b_n - A(b_n - B)\big| \\
&\leq \big|(a_n - A)b_n\big| + \big|A(b_n - B)\big| \\
&= |a_n - A||b_n| + |A||b_n - B| \\
&< \varepsilon' K + |A|\varepsilon' \\
&= \varepsilon.
\end{aligned}$$

Therefore, it is true that $a_n b_n \to AB$, as required.

(iii) Let $\varepsilon > 0$. Since $A \neq 0$, we have that $|A|/2 > 0$, which means that $\varepsilon' := \varepsilon |A|^2/2 > 0$ and we again consider what it means for this sequence to converge in two different ways:

- $a_n \to A$ means there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, we have $|a_n - A| < |A|/2$.
- $a_n \to A$ means there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$, we have $|a_n - A| < \varepsilon'$.

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$,

$$\begin{aligned}
\left|\frac{1}{a_n} - \frac{1}{A}\right| &= \frac{|a_n - A|}{|a_n||A|} \\
&< \frac{\varepsilon'}{|A||A|/2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, it is true that $1/a_n \to 1/A$, as required. □

> **Exercise 7** Prove part (i) of the Algebra of Limits.
>
> [**Hint:** Use an $\varepsilon/2$ argument similar to the one in the proof of Proposition 2.14.]

Example 2.20 We can now use the Algebra of Limits to see that the sequence (a_n) is such that

$$a_n = \frac{2n^2 + n}{n^2 + 7} \to 2.$$

Indeed, we can first re-write this in such a way to use some convergences we already know about (e.g. Exercise 3); this is achieved by dividing through by n^2 to get

$$a_n = \frac{2 + 1/n}{1 + 7/n^2}.$$

We proved $1/n \to 0$ so the Algebra of Limits implies $1/n^2 \to 0$. As a result, we conclude that

$$a_n = \frac{2 + 1/n}{1 + 7/n^2} \to \frac{2+0}{1+0} = 2.$$

Exercise 8 Using the Algebra of Limits, prove that the sequence

$$b_n = \frac{n^4}{2(n+1)^2(n^2+1)} \to \frac{1}{2}.$$

The next result will be very useful to refer to, in order to avoid the difficulties of ε-N proofs.

Theorem 2.21 (Squeeze Rule) *Let (a_n), (b_n) and (c_n) be sequences such that $a_n \to L$ and $c_n \to L$. If $a_n \leq b_n \leq c_n$ for all n, then it follows that $b_n \to L$.*

Proof: Let $\varepsilon > 0$ and consider what it means for these sequences to converge:

- $a_n \to L$ means there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, we have $|a_n - L| < \varepsilon$.
- $c_n \to L$ means there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$, we have $|c_n - L| < \varepsilon$.

In particular, these mean that $a_n > L - \varepsilon$ and $c_n < L + \varepsilon$. Thus, we conclude the following:

$$b_n - L \geq a_n - L > -\varepsilon \qquad \text{and} \qquad b_n - L \leq c_n - L < \varepsilon.$$

Combining these inequalities gives $-\varepsilon < b_n - L < \varepsilon$, equivalent to $|b_n - L| < \varepsilon$, so $b_n \to L$. □

Example 2.22 We exploit the Squeeze Rule to show that the sequence $b_n = \cos(n)/(n^2+1) \to 0$. Indeed, we can take $a_n = -1/n^2$ and $c_n = 1/n^2$, which both converge to zero, and note the bounds $-1 \leq \cos(n) \leq 1$ implies $a_n \leq b_n \leq c_n$. Thus, the Squeeze Rule guarantees that $b_n \to 0$.

Definition 2.23 Consider some real sequence (a_n).

(i) We say (a_n) is increasing if $a_{n+1} \geq a_n$ for all n.
(ii) We say (a_n) is decreasing if $a_{n+1} \leq a_n$ for all n.
(iii) We say (a_n) is strictly increasing if $a_{n+1} > a_n$ for all n.
(iv) We say (a_n) is strictly decreasing if $a_{n+1} < a_n$ for all n.

A sequence is (strictly) monotone if it is either (strictly) increasing or (strictly) decreasing.

Exercise 9 Give an example of a sequence (a_n) which is increasing **and** decreasing. How many other examples are there? Describe them all.

We now develop theory which allows us to state a pseudo-converse to Proposition 2.17 (spoilers for Exercise 6), since we noticed in said exercise that the converse of the result is not true; there exist bounded sequences which do not converge. Indeed, if we add an additional hypothesis to the converse, we do get something which is true. We first need an auxiliary result.

Lemma 2.24 *Let (a_n) be increasing and bounded above. Then, it is convergent.*

Proof: Let $A = \{a_n : n \in \mathbb{Z}^+\}$ be the set of numbers in this sequence. It is clearly non-empty and bounded above, so the Axiom of Completeness guarantees that $\sup(A) = L$ exists. Let $\varepsilon > 0$ be given. Then, $L - \varepsilon < L$ so $L - \varepsilon$ is **not** an upper bound on A. Thus, there exists $N \in \mathbb{Z}^+$ such that $a_N > L - \varepsilon$. That being said, (a_n) is increasing, so $a_n \geq a_N$ for all $n \geq N$. Furthermore, since L is an upper bound on A, we have $a_n \leq L < L + \varepsilon$ for all n. Combining these three inequalities gives that $L - \varepsilon < a_n < L + \varepsilon$, which is to say $|a_n - L| < \varepsilon$, for every $n \geq N$, so we get convergence: $a_n \to L$. □

Lemma 2.25 *Let (a_n) be decreasing and bounded below. Then, it is convergent.*

Exercise 10 Prove Lemma 2.25.

[**Hint:** You can do a direct proof by modifying the proof of Lemma 2.24 **or** you can define a new sequence (b_n) by $b_n = -a_n$ and note that since (a_n) is decreasing, (b_n) is increasing.]

Theorem 2.26 (Monotone Convergence Theorem) *Bounded monotone sequences converge.*

Proof: Bounded means bounded above and bounded below, so Lemmata 2.24 and 2.25 apply. □

Example 2.4 (Revisited) Consider the sequence (a_n) where $a_1 = k$ for any $k \in \mathbb{R}$ and

$$a_{n+1} = \frac{a_n}{1 + a_n^2}.$$

Note that this is slightly more general than the version we encountered in Example 2.4 since we allow the starting value a_1 to be arbitrary. We now have the machinery to prove that this inductively-defined sequence converges for any k. Indeed, it is first clear that the ratio $a_{n+1}/a_n < 1$ for every n, which means that (a_n) is decreasing. Because it is decreasing, it is automatically bounded above by its initial value a_1, but since $a_n > 0$ for all n, it is also bounded below by 0. Hence, (a_n) is a bounded decreasing sequence; the Monotone Convergence Theorem implies that it converges. We can actually compute the limit. Suppose $a_n \to L$, for some $L \in \mathbb{R}$

and define $b_n = a_{n+1}$; it is clear that the sequence $b_n \to L$ but, by the Algebra of Limits applied to the formula for a_{n+1}, we see that $b_n \to L/(1+L^2)$. By the Uniqueness of Limits, we have

$$L = \frac{L}{1+L^2} \qquad \Rightarrow \qquad L = 0.$$

Exercise 11 Let $k \in (0,1)$ be fixed in this interval and define a sequence (a_n) by $a_n = k^n$. Prove that $a_n \to 0$ using the Monotone Convergence Theorem.

Exercise 12 (Harder) Consider the sequence (a_n) where $a_1 = 2$ and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

(i) Prove inductively that $a_n \in [1,2]$ for all n.
(ii) Prove inductively that $a_n^2 \geq 2$ for all n.
(iii) Hence, prove that (a_n) is decreasing.

[**Hint:** Consider the ratio a_{n+1}/a_n.]

(iv) Show that $a_n \to L$ where $L > 0$ such that $L^2 = 2$.

[**Note:** As a consequence, we now know that the (irrational) number $\sqrt{2}$ exists.]

The final part of this section is on so-called *subsequences*; on a basic level, a subsequence is obtained from a sequence by skipping over (possibly infinitely-many) terms in the original sequence. The main goal is to use this theory to avoid the guesswork involved with Definition 2.7, namely we need to 'guess' the correct L to show that a sequence $a_n \to L$.

Definition 2.27 Let (a_n) be a real sequence. A subsequence is a sequence (b_k) where there exists a strictly increasing sequence (n_k) of positive integers such that $b_k = a_{n_k}$.

Example 2.28 Here are some examples and non-examples of subsequences.

(i) Let (a_n) be **any** sequence; (a_k) is a subsequence of itself, where $n_k = k$.

(ii) Let (a_n) be **any** sequence; $(b_k) = (a_{m+1}, a_{m+2}, \ldots)$ is a subsequence, where $n_k = m + k$.

(iii) Let $(a_n) = (-1, 1, -1, 1, \ldots)$; $(b_k) = (1, 1, 1, 1, \ldots)$ is a subsequence, where $n_k = 2k$.

(iv) Let $(a_n) = (1, 1/2, 1/3, \ldots)$; $(b_k) = (1/2, 1/4, 1/8, \ldots)$ is a subsequence, where $n_k = 2^k$.

(v) Let $(a_n) = (1, 2, 3, 4, \ldots)$; $(b_k) = (2, 1, 3, 4, \ldots)$ is **not** a subsequence.

Theorem 2.29 *Let (a_n) be such that $a_n \to L$. If (b_k) is a subsequence, then $b_k \to L$.*

Proof: Let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. Because (b_k) is a subsequence, it is of the form $b_k = a_{n_k}$ where (n_k) is a strictly increasing sequence of positive integers. Obviously, $n_1 \geq 1$ but, more than that, if we assume that $n_k \geq k$, then we can conclude $n_{k+1} \geq n_k + 1 \geq k + 1$; we have really just done a stealthy induction argument to show that $n_k \geq k$ for all k. Therefore, for all $k \geq N$, we see that

$$n_k \geq n_N \geq N \qquad \Rightarrow \qquad |b_k - L| = \left|a_{n_k} - L\right| < \varepsilon. \qquad \square$$

Example 2.13 (Revisited) We can now give a much slicker proof that the sequence defined by $a_n = (-1)^n$ diverges, as opposed to messing about with ε-N arguments. Indeed, we see that this sequence has the following subsequences:

$$(a_{2k}) = (1, 1, 1, 1, ...) \qquad \text{and} \qquad (a_{2k+1}) = (-1, -1, -1, -1, ...).$$

These are constant sequences, so we see that $a_{2k} \to 1$ and $a_{2k+1} \to -1$. Assume to the contrary that $a_n \to L$ for some $L \in \mathbb{R}$. By Theorem 2.29, it must be that all subsequences converge to this limit also, so $1 = L = -1$, a contradiction.

Lemma 2.30 (Tail Lemma) *Let (a_n) be a sequence and define the subsequence (b_n) by $b_n = a_{n+m}$ for some $m \in \mathbb{Z}^+$, i.e. remove the first m terms. If $b_n \to L$, then $a_n \to L$.*

Proof: Let $\varepsilon > 0$ be given. Then, there exists $K \in \mathbb{Z}^+$ such that $|b_n - L| < \varepsilon$ for all $n \geq K$. Now, define $N = K + m \in \mathbb{Z}^+$. Then, for all $n \geq N$, which is equivalent to all $n - m \geq K$,

$$|a_n - L| = |b_{n-m} - L| < \varepsilon. \qquad \square$$

In simple terms, the Tail Lemma allows us to remove a troublesome start to a sequence; if we can say something about the convergence of 'the rest of' a sequence, then we now know that putting back the cut-off points will not change the convergence.

Example 2.31 Consider the sequence given by $a_n = n^3/2^n$. Ideally, we use the Monotone Convergence Theorem, but we can convince ourselves this sequence isn't monotone. However,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{2n^3} = \frac{n^3 + 3n^2 + 3n + 1}{2n^3} \leq \frac{n^3 + 3n^2 + 3n^2 + n^2}{2n^3} = \frac{1}{2} + \frac{7}{2n}.$$

Now, $1/2 + 7/2n < 1$ for $n > 7$ (equivalent to $n \geq 8$ since we are working with positive integers).

Thus, the 'tail' defined by $b_n = a_{n+7}$ is decreasing and bounded (above by b_1 and below by 0) so the Monotone Convergence Theorem implies that (b_n) converges, to which the Tail Lemma applies, giving that (a_n) converges.

Definition 2.32 Let (a_n) be a real sequence. We say a term a_m is dominant if every subsequent term is not larger than it, that is to say $a_n \leq a_m$ for all $n > m$.

Lemma 2.33 *Every sequence has a monotone subsequence.*

Proof: Let (a_n) be a sequence and D the set of dominant terms. There are two cases to consider.

(i) If $|D| = \infty$, then the subsequence consisting of the dominant terms is decreasing, by definition of dominant. Thus, (a_n) has a decreasing subsequence.

(ii) If $|D| < \infty$ (or if $D = \emptyset$), then there exists a term a_m, say, beyond which there are **no** dominant terms. Let $n_1 = m+1$; since a_{n_1} is **not** dominant, there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$, but since a_{n_2} is **not** dominant, there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$, etc. In this way, we get a strictly increasing sequence of positive integers $(n_1, n_2, n_3, ...)$ such that $a_{n_{k+1}} > a_{n_k}$ for all k. Thus, (a_{n_k}) is an increasing subsequence. □

Theorem 2.34 (Bolzano-Weierstrass Theorem) *Every bounded sequence has a convergent subsequence.*

Exercise 13 Prove the Bolzano-Weierstrass Theorem.

We now reach the point where we develop important theory which allows us to circumvent 'knowing' what the limit L of a sequence (a_n) is before actually proving $a_n \to L$ rigorously. This is known as the *Cauchy property*.

Definition 2.35 A real sequence (a_n) is Cauchy if, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, we have $|a_n - a_m| < \varepsilon$.

Remark 2.36 The Cauchy property is awfully similar to Definition 2.7 with a key difference; no mention of a real number L. Instead, we look at the difference between two terms a_n and a_m. In words, where convergence is about having all terms after a certain point being within distance ε of the limit L, the Cauchy property is about having all terms after a certain point being within distance ε of each other.

Exercise 14 Prove that the sequence (a_n) given by $a_n = 1 + 1/n$ is Cauchy.

Proposition 2.37 *If (a_n) is convergent, then it is Cauchy.*

Proof: Assume $a_n \to L$ and let $\varepsilon > 0$. Then, there exists $N \in \mathbb{Z}^+$ such that, for all $n, \geq N$, we have $|a_n - L| < \varepsilon/2$. Then, for all $n, m \geq N$,

$$\begin{aligned} |a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |a_m - L| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

□

Lemma 2.38 *If (a_n) is Cauchy, then it is bounded.*

Proof: Let (a_n) be Cauchy. Then, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, we have $|a_n - a_m| < 1$. In particular, we have $|a_n| < 1 + |a_N|$. Similar to the proof of Proposition 2.17, we note that $|a_n|$ is bounded at least for all $n \geq N$, so we need to consider the maximum of this upper bound with the absolute values of the points that came before, namely

$$M = \max\{\, |a_1|, \ldots, |a_{N-1}|, 1 + |a_N|\},$$

which certainly exists. Thus, it is clear that $|a_n| \leq M$ for all n, as required. □

Lemma 2.39 *If (a_n) is Cauchy and it has a subsequence $a_{n_k} \to L$, then $a_n \to L$.*

Proof: Let $\varepsilon > 0$ and consider both the convergence and the Cauchy property:

- $a_{n_k} \to L$ means there is $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, we have $\left|a_{n_k} - L\right| < \varepsilon/2$.
- (a_n) Cauchy means there is $N_2 \in \mathbb{Z}^+$ such that, for all $n, m \geq N_2$, we have $|a_n - a_m| < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$,

$$\begin{aligned} |a_n - L| &= \left|a_n - a_{n_N} + a_{n_N} - L\right| \\ &\leq \left|a_n - a_{n_N}\right| + \left|a_{n_N} - L\right| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

□

Theorem 2.40 *A sequence (a_n) is convergent if and only if it is Cauchy.*

Proof: If (a_n) converges, then it is Cauchy (Proposition 2.37). Conversely, if (a_n) is Cauchy, it is bounded (Lemma 2.38), so it has a convergent subsequence (Bolzano-Weierstrass Theorem), which means it converges (Lemma 2.39). □

Exercise 15 Prove directly from Definition 2.35 that the sum of two Cauchy sequences is Cauchy. Can we say anything about the product of two Cauchy sequences? What about the reciprocal of a (non-zero) Cauchy sequence?

3 Series

We have looked at some examples of *finite series* before, notably when looking at proof by induction. However, a *series* really means an infinite sum. We will give a rigorous definition and discuss what it means for a series to converge.

Definition 3.1 A real series is an infinite sum of the form $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$. We call $s_k = \sum_{n=1}^{k} a_n = a_1 + \cdots + a_k$ the k^{th} partial sum of the series.

Definition 3.2 A real series $\sum_{n=1}^{\infty} a_n$ converges to a number $L \in \mathbb{R}$ if the sequence of partial sums (s_k) converges in the usual sense, that is for each $\varepsilon > 0$, there exists $K \in \mathbb{Z}^+$ such that, for all $k \geq K$, we have $|s_k - L| < \varepsilon$. Otherwise, we say the real series diverges.

Example 3.3 Consider the series $\sum_{n=1}^{\infty} 1/n(n+1)$. We can re-write the summand as the difference of two fractions, from which we can see that the k^{th} partial sum is as follows:

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \sum_{n=1}^{k} \frac{1}{n} - \sum_{n=1}^{k} \frac{1}{n+1} \\ &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} \right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1} \right) \\ &= 1 - \frac{1}{k+1} \\ &\to 1 \end{aligned}$$

by the Algebra of Limits; the sum converges.

Proposition 3.4 (Harmonic Series) *The series* $\sum_{n=1}^{\infty} 1/n$ *diverges.*

Proof: We consider the $2^{p\text{th}}$ partial sum as follows:

$$\begin{aligned} s_{2^p} &= 1 + \frac{1}{2} + \cdots + \frac{1}{2^p} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{p-1}-1} + \cdots + \frac{1}{2^p}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^p} + \cdots + \frac{1}{2^p}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= 1 + \frac{p}{2}. \end{aligned}$$

Hence, the subsequence (s_{2^p}) is unbounded above, so the Bolzano-Weierstrass Theorem implies that it diverges. Consequently, the Harmonic Series diverges. □

Proposition 3.5 (Geometric Series) *The series* $\sum_{n=0}^{\infty} \alpha^n = 1/(1-\alpha)$ *for* $|\alpha| < 1$.

Proof: We consider the k^{th} partial sum as follows:

$$\begin{aligned} && s_k &= 1 + \alpha + \alpha^2 + \cdots + \alpha^k \\ \Rightarrow && \alpha s_k &= \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{k+1} \\ \Rightarrow && (1-\alpha) s_k &= 1 - \alpha^{k+1} \\ \Rightarrow && s_k &= \frac{1-\alpha^{k+1}}{1-\alpha}. \end{aligned}$$

This proves Proposition **??**. Finally, Exercise 11 implies $\alpha^{k+1} \to 0$ and so $s_k \to 1/(1-\alpha)$. □

4 Continuous Functions

5 Differentiation

6 Integration

7 Basic Functional Analysis

8 Exercise Solutions

We provide detailed solutions to the exercises interwoven within each section of the module. Hopefully you have given these questions a try whilst on your learning journey with the module. But mathematics is difficult, so don't feel disheartened if you had to look up an answer before you knew where to begin (we have all done it)!

Solutions to Exercises in Section 2

Exercise 1 Find the first n with $c_n = 1$ to five decimal places, for (c_n) in Example 2.2.

[**Hint:** This means that $0.999995 \leq c_n < 1.000005$, so simply rearrange this for n.]

Solution: Per the hint, we have that $0.999995 \leq 1 + 5/n^2 < 1.000005$ which is to say that

$$-0.00001 \leq \frac{1}{n^2} \leq 0.000001 \quad \Rightarrow \quad n^2 > 1\,000\,000,$$

where we ignore the negative part since $n \in \mathbb{Z}^+$. It follows from this that $n > 1000$, so the first such approximation of one to give decimal places occurs at $n = 1001$. □

Exercise 2 Explain the properties $x \leq |x|$ and $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.

Solution: If $x \leq 0$, then $x = |x|$, whereas if $x < 0$, then $x < |x|$ since $|x|$ is non-negative by definition. Combining these two cases gives us that $x \leq |x|$. Next, we see that $xy \geq 0$ precisely when both $x, y \geq 0$ or both $x, y < 0$. On the other hand, $xy < 0$ precisely when one of $x < 0$ and $y < 0$. Therefore, $|xy| = |x||y|$ is a quick consequence of these cases. □

Exercise 3 Give a direct ε-N proof that the sequence (a_n) where $a_n = 1/n$ converges.

[**Hint:** You must first suggest what the limit L is and then go ahead with the proof.]

Solution: (Rough Work) We should suggest that $a_n \to 0$. Intuitively, when n is large, we will see that $1/n$ becomes very small. Since $n > 0$, it follows that $1/n > 0$, but it will get smaller and smaller, so a 'good guess' at the limit is indeed $L = 0$. Now, we need to estimate the quantity $|a_n - 1|$ in order to be able to determine what our $N \in \mathbb{Z}^+$ will depend on:

$$|a_n - 0| \quad = \quad \big|1/n - 0\big| \quad = \quad \big|1/n\big| \quad = \quad 1/n \quad \leq \quad 1/N,$$

whenever $n \geq N$. Therefore, to ensure that $1/N \leq \varepsilon$, we must pick $N > 1/\varepsilon$.

(Proof) Let $\varepsilon > 0$ be given. There is $N \in \mathbb{Z}^+$ with $N > 1/\varepsilon$ by the Archimedean Property of $\mathbb{R}$. Then, for all $n \geq N$, it follows that

$$\begin{aligned} |a_n - 0| &= \big|1/n - 0\big| \\ &= \big|1/n\big| \\ &= 1/n \\ &\leq 1/N \\ &< \varepsilon. \end{aligned}$$

Therefore, we can conclude that $a_n \to 0$. □

Note: We will often miss out the rough work part in future when writing a direct ε-N proof (but, of course, this is still how we got to the solutions that are presented here).

Exercise 4 Give a direct ε-N proof that the so-called *constant sequence* (b_n) converges, that is the sequence is defined by $b_n = k$ for all n, where $k \in \mathbb{R}$ is some fixed number.

Solution: Let $\varepsilon > 0$ be given. Then, choosing $N = 1$, for all $n \geq 1$, it follows that

$$\begin{aligned} |b_n - k| &= |k - k| \\ &= 0 \\ &< \varepsilon. \end{aligned}$$

Therefore, we can conclude that $b_n \to k$. □

Exercise 5 Prove that the sequence (a_n) given by $a_n = n$ is divergent.

Solution: Assume to the contrary that $a_n \to L$ for some $L \in \mathbb{R}$. Then, this means that for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. In particular, this should work for $\varepsilon = 1/2$, meaning that $|a_n - L| < 1/2$ for every $n \geq N$. In particular then,

$$|a_N - L| = |N - L| < \frac{1}{2} \quad \text{and} \quad |a_{N+2} - L| = |N + 2 - L| < \frac{1}{2}.$$

However, we can see that

$$\begin{aligned}2 &= N + 2 - N \\ &= |N + 2 - N| \\ &= |N + 2 - L + L - N| \\ &\leq |N + 2 - L| + |N - L| \\ &< 1/2 + 1/2 \\ &= 1,\end{aligned}$$

a clear contradiction. □

Exercise 6 State the converse to Proposition 2.17 and determine if it is true.

Solution: This is the converse: "if a sequence is bounded, then it is convergent". It is not true. Indeed, Example 2.13 provides us with a counterexample, because the sequence $a_n = (-1)^n$ is clearly bounded above by 1 and below by -1, but we have proved that (a_n) diverges. □

Exercise 7 Prove part (i) of the Algebra of Limits.

[**Hint:** Use an $\varepsilon/2$ argument similar to the one in the proof of Proposition 2.14.]

Solution: Let $\varepsilon > 0$. We consider what it means for these sequences to converge separately:

- $a_n \to A$ means there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, we have $|a_n - A| < \varepsilon/2$.
- $b_n \to B$ means there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$, we have $|b_n - B| < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$,

$$\begin{aligned}\big|a_n + b_n - (A + B)\big| &= |a_n - A + b_n - B| \\ &\leq |a_n - A| + |b_n - B| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon.\end{aligned}$$

Therefore, it is true that $a_n + b_n \to A + B$. □

Exercise 8 Using the Algebra of Limits, prove that the sequence

$$b_n = \frac{n^4}{2(n+1)^2(n^2+1)} \to \frac{1}{2}.$$

Solution: First, we can expand the denominator and then divide through by n^4 to see that

$$b_n = \frac{n^4}{2n^4 + 4n^3 + 4n^2 + 4n + 2} = \frac{1}{2 + 4/n + 4/n^2 + 4/n^3 + 2/n^4},$$

so by the Algebra of Limits, we have that $1/n^k \to 0$ for any $k \geq 1$ and so each of the fractions in the denominator are zero in the limit. Consequently, $b_n \to 1/2$, as required. □

Exercise 9 Give an example of a sequence (a_n) which is increasing **and** decreasing. How many other examples are there? Describe them all.

Solution: A sequence is increasing and decreasing precisely when $a_{n+1} \leq a_n \leq a_{n+1}$ for all n, that is $a_n = a_{n+1}$ for all n. The only such sequences satisfying this are constant sequences. As for a particular example, the sequence (a_n) defined by $a_n = \pi^e/200!$ will do. □

Exercise 10 Prove Lemma 2.25.

[**Hint:** You can do a direct proof by modifying the proof of Lemma 2.24 **or** you can define a new sequence (b_n) by $b_n = -a_n$ and note that since (a_n) is decreasing, (b_n) is increasing.]

Solution: (Direct) Let $A = \{a_n : n \in \mathbb{Z}^+\}$ be the set of numbers in this sequence. It is clearly non-empty and bounded below, so Proposition **??** guarantees that $\inf(A) = K$ exists. Let $\varepsilon > 0$ be given. Then, $K + \varepsilon > K$ so $K + \varepsilon$ is not a lower bound on A. Thus, there exists $N \in \mathbb{Z}^+$ such that $a_N < K + \varepsilon$. That being said, (a_n) is decreasing, so $a_n \leq a_N$ for all $n \geq N$. Furthermore, since K is a lower bound on A, we have $a_n \geq K > K - \varepsilon$ for all n. Combining these three inequalities gives that $K - \varepsilon < a_n < K + \varepsilon$, which is to say $|a_n - K| < \varepsilon$, for every $n \geq N$, so we get convergence: $a_n \to K$.

(Easy) Let (a_n) be decreasing and bounded below by K. The sequence (b_n) defined by $b_n = -a_n$ is increasing and bounded above by $-K$. By Lemma 2.24, (b_n) is convergent; the Algebra of Limits implies that (a_n) is convergent. □

Exercise 11 Let $k \in (0,1)$ be fixed in this interval and define a sequence (a_n) by $a_n = k^n$. Prove that $a_n \to 0$ using the Monotone Convergence Theorem.

Solution: It is obvious that $a_n > 0$ for all n. Furthermore, it is clear that $a_{n+1} = ka_n < a_n$ because $k \in (0,1)$. Therefore, (a_n) is a (Strictly) decreasing sequence which is bounded below. The Monotone Convergence Theorem implies that it converges to some limit, say $a_n \to L$. We now define a new sequence (b_n) by $b_n = a_{n+1}$, that is we ignore the first term in (a_n), from which it's trivial that $b_n \to L$ also. However, we can also write $b_n = ka_n$, to which we apply the Algebra of Limits to get $b_n \to kL$. By the Uniqueness of Limits, it must be that $L = kL$, and so $L = 0$ is the only option. We therefore conclude that $a_n \to 0$. □

Exercise 12 (Harder) Consider the sequence (a_n) where $a_1 = 2$ and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right).$$

(i) Prove inductively that $a_n \in [1,2]$ for all n.
(ii) Prove inductively that $a_n^2 \geq 2$ for all n.
(iii) Hence, prove that (a_n) is decreasing.

[**Hint:** Consider the ratio a_{n+1}/a_n.]

(iv) Show that $a_n \to L$ where $L > 0$ such that $L^2 = 2$.

[**Note:** As a consequence, we now know that the (irrational) number $\sqrt{2}$ exists.]

Solution: (i) The base case $n = 1$ is clear. Suppose now that $a_k \in [1,2]$ for some $k \in \mathbb{Z}^+$. Thus,

$$a_{k+1} = \frac{1}{2}\left(a_k + \frac{2}{a_k}\right) \geq \frac{1}{2}\left(1 + \frac{2}{2}\right) = 1 \quad \text{and} \quad a_{k+1} = \frac{1}{2}\left(a_k + \frac{2}{a_k}\right) \leq \frac{1}{2}\left(2 + \frac{2}{1}\right) = 2,$$

where we have used the fact $a_k \leq 2 \Leftrightarrow 1/a_k \geq 1/2$ and $a_k \geq 1 \Leftrightarrow 1/a_k \leq 1$. Therefore, $a_{k+1} \in [1,2]$ and the result holds for all n by the principal of mathematical induction.

(ii) The base case $n = 1$ is clear. Assume to the contrary $a_k^2 < 2$ for some integer $k \geq 2$, i.e.

$$\begin{aligned} & \frac{1}{4}\left(a_{k-1}^2 + 4 + \frac{4}{a_{k-1}^2}\right) < 2 \\ \Rightarrow \quad & a_{k-1}^4 + 4a_{k-1}^2 + 4 < 8a_{k-1}^2 \\ \Rightarrow \quad & a_{k-1}^4 - 4a_{k-1}^2 + 4 < 0 \end{aligned}$$

$$\Leftrightarrow \qquad (a_{k-1}^2 - 2)^2 < 0,$$

which is a contradiction because the left-hand-side is a square and must be non-negative. Thus, by the principal of mathematical induction, the result holds for all n.

(iii) Because $a_n^2 \geq 2$ for all n by (ii), we see that

$$\frac{a_{n+1}}{a_n} = \frac{1}{2} + \frac{2}{a_n^2} \leq \frac{1}{2} + \frac{1}{2} = 1,$$

which is to say $a_{n+1} \leq a_n$, meaning the sequence is decreasing.

(iv) Finally, because (a_n) is bounded by (i) and decreasing by (iii), the Monotone Convergence Theorem implies that $a_n \to L$ for some $L \in \mathbb{R}$. But then, the sequence missing the first term $a_{n+1} \to L$ also. Hence, applying the Algebra of Limits to the formula for a_{n+1} means that

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right) \qquad \Rightarrow \qquad L^2 = 2.$$

Because $a_n \geq 1$ for all n, it must be that $L \geq 1 > 0$, allowing us to conclude that $L = \sqrt{2}$. □

Note: We can generalise the sequence in Exercise 12 to show that the square root of any positive number exists. Indeed, we can consider the sequence (a_n) where $a_1 = \beta > 0$ and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{\beta}{a_n}\right).$$

We can proceed as above to show that $a_n \to L$ where $L > 0$ such that $L^2 = \beta$. Not only does this prove that square roots of positive real numbers exist, but that the limit of a sequence of rational numbers (as $\mathbb{Q}$ is closed under division by non-zeros) can be irrational!

Exercise 13 Prove the Bolzano-Weierstrass Theorem.

Solution: Let (a_n) be a bounded sequence. By Lemma 2.33, it has a monotone subsequence (a_{n_k}), which is clearly bounded. By the Monotone Convergence Theorem, (a_{n_k}) converges. □

Exercise 14 Prove that the sequence (a_n) given by $a_n = 1 + 1/n$ is Cauchy.

Solution: Let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{Z}^+$ such that $N > 2/\varepsilon$ by the Archimedean

Property of $\mathbb{R}$. Then, for all $n, m \geq N$, it follows that

$$\begin{aligned} |a_n - a_m| &= \big|1 + 1/n - (1 + 1/m)\big| \\ &= \big|1/n - 1/m\big| \\ &\leq \big|1/n\big| + \big|1/m\big| \\ &= 1/n + 1/m \\ &\leq 1/N + 1/N \\ &= 2/N \\ &< \varepsilon. \end{aligned}$$

Therefore, (a_n) is Cauchy. □

> **Exercise 15** Prove directly from Definition 2.35 that the sum of two Cauchy sequences is Cauchy. Can we say anything about the product of two Cauchy sequences? What about the reciprocal of a (non-zero) Cauchy sequence?

Solution: Let $\varepsilon > 0$ be given. We consider what it means for two sequences to be Cauchy:

- (a_n) Cauchy means there is $N_1 \in \mathbb{Z}^+$ such that, for all $n, m \geq N_1$, we have $|a_n - a_m| < \varepsilon/2$.
- (b_n) Cauchy means there is $N_2 \in \mathbb{Z}^+$ such that, for all $n, m \geq N_2$, we have $|b_n - b_m| < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then, for all $n, m \geq N$,

$$\begin{aligned} \big|a_n + b_n - (a_m + b_m)\big| &= |a_n - a_m + b_n - b_m| \\ &\leq |a_n - a_m| + |b_n - b_m| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Therefore, it is true that $(a_n) + (b_n)$ is Cauchy. By Theorem 2.40, we know that convergence and the Cauchy property are equivalent (for sequences in $\mathbb{R}$), so the Algebra of Limits can be translated into the language of Cauchy sequences: the product of two Cauchy sequences is Cauchy and the reciprocal of a (non-zero) Cauchy sequence is Cauchy. □