Introduction to Geometry

Prison Mathematics Project

Introduction

Hello and welcome to the module on Introduction to Geometry! What follows is a module intended to support the reader in learning this fascinating topic. The Prison Mathematics Project (PMP) realises that you may be practising mathematics in an environment that is highly restrictive, so this text can both be used independently and does not require a calculator.

What is Geometry?

Geometry is one of the most ancient branches of mathematics. One of the earliest recorded instances of humans studying geometry comes from the second millennium BC. Ancient Greece was a haven for geometers, including Thales, Pythagoras and Archimedes. Perhaps the most influential of the time was Euclid, whose series of thirteen books, the *Elements*, introduced mathematical rigour through an axiomatic system. It is the earliest format of the system of axioms, definitions, theorems and proofs that we use today, see Chapter **??** for instance. Modern geometry has been developed by many a mathematician, including Gauss, Riemann and Poincaré.

Learning in this Module

The best way to learn mathematics is to do mathematics. Indeed, education isn't something that happens more than it is something we should all participate in. You will find various exercise questions and worked examples in these notes so that you may try to solve problems and deepen your understanding of this topic. Although the aim is for everything to only require the content of this module, you are encouraged to use any other sources you have at your disposal.

Acknowledgements

These notes are based on a lecture course by B. Marsh at the University of Leeds.

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1 Preliminaries

2 Axiomatic Two-Dimensional Geometry

The first port of call is two-dimensional geometry, which will be discussed from both an axiomatic and a coordinate point-of-view. Indeed, we begin with the former approach.

Note: A line is infinitely long but a line segment has finite length, connecting two points.

Now that we have cleared up the difference, we can make a start on the axiomatic approach to plane geometry (where the word plane is the word we use to refer to a flat two-dimensional surface which extends infinitely far – think of what a pair of (x, y)-axes form for a mathematical example or an infinitely-large chess board for a 'real world' example).

Definition 2.1 A right angle is defined as one of the angles formed when a line segment ends on a line, forming two equal angles; this is pictured in Figure 1. In this case, the line and line segment are called perpendicular (or orthogonal).

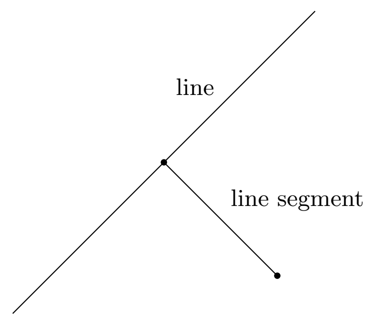

Figure 1: The geometric picture of a right angle per Definition 2.1.

Definition 2.2 We say that two lines are parallel if they do **not** intersect each other.

Axiom 2.3 (Euclidean Axioms)

E1. *There is a unique line segment between any two distinct points.*

E2. *Any line segment can be extended to a line.*

E3. *There is a circle at point P with radius r, for any P and length r line segment at P.*

E4. *All right angles are equal.*

E5. *For a line L and a point P* **not** *on L, there is a unique line through P parallel to L.*

Remark 2.4 The axiom E5 is the Parallel Postulate. There was a time where mathematicians did not know if it was redundant: could E5 be deduced from the others? The answer is **no**. In fact, if we remove E5, we get more exotic forms of geometry (hyperbolic/elliptic geometry).

Throughout, we will label our line segments by the endpoints, meaning if a segment connects A and B, it will be denoted AB. We do this also for the line we get by extending the ray AB.

Definition 2.5 Let L be a line segment containing a point P. Then, a ray from P is the subset which starts at P and extends infinitely along the rest of the line, as in Figure 2.

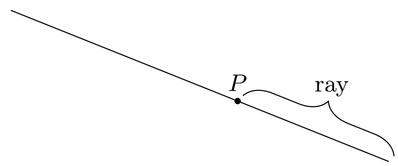

Figure 2: The geometric picture of a ray per Definition 2.5.

Definition 2.6 Let A, B, C be points. They are collinear if there exists a line containing all of them. Otherwise, they are called non-collinear.

Exercise 1 We discuss three points in Definition 2.6, but what can we say about the (simpler) situation of only two points: are two points always collinear? Are they always non-collinear? Give a one-sentence proof of the correct statement by using Axiom 2.3.

In Definition 2.7, and throughout these notes, we use the measurement of *radians* for angles, not *degrees*, unless stated otherwise. There is a quick conversion, however: degrees to radians can be done by multiplying by $\pi/180$ and radians to degrees can be done by multiplying by $180/\pi$.

Definition 2.7 Let A, B, C be non-collinear points. The non-reflexive angle between the rays AB and AC is a number $\angle ABC \in (0, \pi)$.

We can extend Definition 2.7 as follows:

- If B is between A and C and they **are** collinear, we set $\angle ABC = \pi$.
- If B is **not** between A and C and they **are** collinear, we set $\angle ABC = 0$.

Definition 2.8 Let A, B, C be non-collinear points. The triangle ABC is the shape formed with line segments AB, BC, CA. The points A, B, C are then called the vertices of the triangle and the aforementioned line segments are the edges of the triangle.

Remark 2.9 Strictly speaking, we should say that the points A, B, C are the triangle in Definition 2.8; the line segments AB, BC, CA should be called *trilateral*, i.e. they connect the points of a triangle (there is a slightly more technical and general meaning but this won't be covered here).

Definition 2.10 We say that two triangles ABC and $A'B'C'$ are congruent if there is a bijective correspondence between the vertices such that corresponding edges are equal in length and corresponding angles are equal in size. This is denoted by $ABC \simeq A'B'C'$.

Example 2.11 Consider the triangles in Figure 3. We will detail the congruence between them.

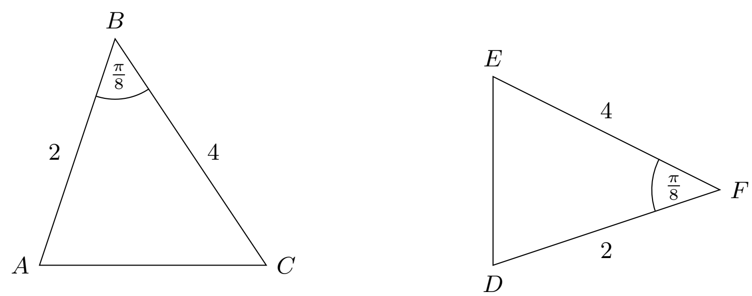

Figure 3: Two triangles ABC and DEF (not drawn to scale).

Immediately, we know that our congruence (bijection) needs to be such that $B \mapsto F$ in order to preserve the angle at that vertex. Then, to preserve the lengths of the sides, we see that $A \mapsto D$ and $C \mapsto E$. In practice, we will often immediately relabel the second triangle so that $D \coloneqq A'$, $F \coloneqq B'$ and $E \coloneqq C'$. This makes clear that the triangles are congruent.

Exercise 2 Prove that triangle congruence is an equivalence relation.

Fortunately, there are two useful axioms from which we can quickly determine congruences.

Axiom 2.12 (Side-Side-Side) *If $AB = A'B'$ and $BC = B'C'$ and $CA = C'A'$, then $ABC \simeq A'B'C'$; in words, triangles with three common side lengths are congruent.*

Axiom 2.13 (Side-Angle-Side) *If $AB = A'B'$ and $BC = B'C'$ and $\angle ABC = \angle A'B'C'$, then $ABC \simeq A'B'C'$; in words, triangles with two common side lengths and a common angle between said common sides are congruent.*

Note: Throughout, we refer to Axiom 2.12 as SSS and to Axiom 2.13 as SAS.

Proposition 2.14 (Angle-Side-Angle) *IF $AB = A'B'$ and $\angle CBA = \angle C'B'A'$ and $\angle CAB = \angle C'A'B'$, then $ABC \simeq A'B'C'$; in words, triangles with one common side length and two common angles at either end of the common side are congruent.*

Proof: Suppose first that $AC \geq A'C'$ and let X be a point on the line segment AC such that $AX = A'C'$ (because AC is assumed equal to or longer than $A'C'$, we find the point X along it at which we have a length exactly equal to $A'C'$). We will consult Figure 4 to prevent notation from clouding the situation; note that the single-marked angles are equal and the double-marked angles are equal. The perpendicular mark on the side lengths means that they are equal.

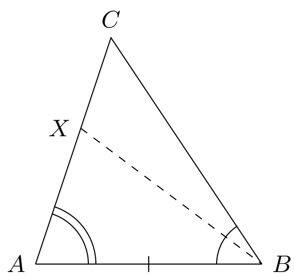

Figure 4: The construction for the proof of Angle-Side-Angle.

By definition of X, we have that $AX = A'C'$, but given that $AB = A'B'$ and $\angle BAX = \angle B'A'C'$, we can conclude from SAS that $ABX \simeq A'B'C'$. Hence, $\angle XBA = \angle C'B'A'$, but this is just $\angle CBA$ from what we assume in the statement. Consequently, we must have $X = C$, from which we see that $ABX = ABC \simeq A'B'C'$. An identical argument works if $AC < A'C'$. □

Note: Similarly to the previous note, we refer to Proposition 2.14 as ASA.

Exercise 3 We have seen that the conditions SSS, SAS and ASA guarantee congruence. Can we conclude the same about 'AAA', that is does having all three angles in common imply that two triangles are congruent? If so, prove it. If not, give a counterexample.

Example 2.11 (Revisited) Looking back at Example 2.11, we are now equipped with two useful axioms and one useful proposition which allow us to demolish the task of determining congruence with relative ease. Indeed, we can immediately see that $ABC \simeq DEF$ as a result of SAS.

Definition 2.15 We call two triangles ABC and $A'B'C'$ similar if there is a bijective correspondence between the vertices such that the corresponding angles are equal in size. This is denoted $ABC \sim A'B'C'$.

Remark 2.16 It is clear that, by comparing Definition 2.15 to Definition 2.10, a key assumption is missing: similar triangles need not preserve the lengths of edges. So, congruent $\Rightarrow$ similar.

Example 2.17 As per Remark 2.16, any pair of congruent triangles is automatically similar. For an example of two similar triangles that are **not** a congruence, see Figure 5. Indeed, here we see that $ABC \sim DEF$ but where $ABC \not\cong DEF$. Hence, the implication in Remark 2.16 is one-way only. Again, we often relabel the second triangle so that $D := A'$, $E := B'$ and $F := C'$.

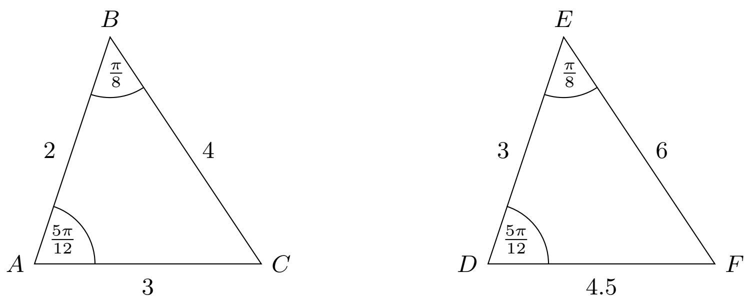

Figure 5: Two similar (but incongruent) triangles.

That said, there is a way to determine similarity based entirely on the lengths of the sides (or, more accurately, the ratios of the lengths between corresponding triangles).

Proposition 2.18 *Let ABC and $A'B'C'$ be triangles. They are similar if and only if*

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}.$$

In words, Proposition 2.18 tells us that two triangles are similar if and only if the ratios of corresponding sides is the same irrespective of the pair of sides chosen. This ratio is called the scale factor and it is the constant that multiplies to transform the lengths in ABC to the lengths in $A'B'C'$.

Definition 2.19 Let L_1 and L_2 be distinct lines. A line K is called a transversal of L_1 and L_2 if it crosses them. We use Figure 6 for the remainder of the definition.

- The pairs of vertically-opposite angles are (α_1, γ_1), (β_1, δ_1), (α_2, γ_2), (β_2, δ_2).
- The pairs of corresponding angles are (α_1, α_2), (β_1, β_2), (γ_1, γ_2), (δ_1, δ_2).
- The pairs of alternate angles are (α_2, γ_1), (β_1, δ_2).

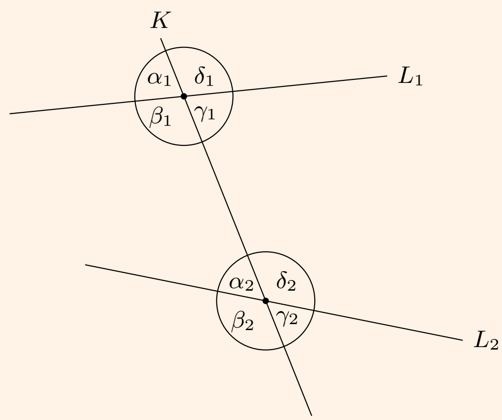

Figure 6: A transversal K of lines L_1 and L_2.

We wish to prove things about the angles at the intersections of a transversal of two lines, as in Figure 6, and how knowing about the angles can inform us about the parallelity of L_1 and L_2.

Definition 2.20 Consider two angles θ and φ.

- They are complementary if $\theta + \varphi = \pi/2$.
- They are supplementary if $\theta + \varphi = \pi$.

Proposition 2.21 *Vertically-opposite angles are equal.*

Proof: Consider Figure 6. It is clear that the pairs (α_1, β_1) and (β_1, γ_1) are each supplementary (angles on a line sum to π, from the extension to Definition 2.7). In other words, $\alpha_1 + \beta_1 = \pi$ and $\beta_1 + \gamma_1 = \pi$. If we subtract the second equation from the first, we see that $\alpha_1 - \gamma_1 = 0$, which is to say that $\alpha_1 = \gamma_1$. The same proof works for any vertically-opposite angles. □

Lemma 2.22 *Let L_1 and L_2 be distinct lines and K a transversal. If L_1 and L_2 are parallel, then the alternate angles are equal.*

Proof: Again using Figure 6, assume to the contrary that $\beta_1 \neq \delta_2$. In particular, this means one of the angles will be greater than the other. Without loss of generality, suppose that $\beta_1 > \delta_2$.

Because angles on a line are supplementary, we know that $\beta_1 + \gamma_1 = \pi$, which implies that $\delta_2 + \gamma_1 < \pi$. But now, lines extending from angles which sum to less than π must meet, i.e. L_1 and L_2 are not parallel, a contradiction. □

Theorem 2.23 *Let L_1 and L_2 be distinct lines and K a transversal. Then, L_1 and L_2 are parallel if and only if the alternate angles are equal.*

Proof: ($\Rightarrow$) This is Lemma 2.22.

($\Leftarrow$) Suppose that all alternate angles are equal. In particular, this means that $\alpha_2 = \gamma_1$. By the Parallel Postulate, there exists a line through the intersection point P of the lines K and L_1 which is parallel to L_2; call this line L_3. Because L_2 and L_3 are parallel, by construction, Lemma 2.22 implies that $\alpha_2 = \gamma_3$ (since they are alternate), where γ_3 is as in Figure 7. Combining this with the first equation implies that $\gamma_1 = \gamma_3$, which is equivalent to $L_1 = L_3$. Thus, L_1 and L_2 are parallel. □

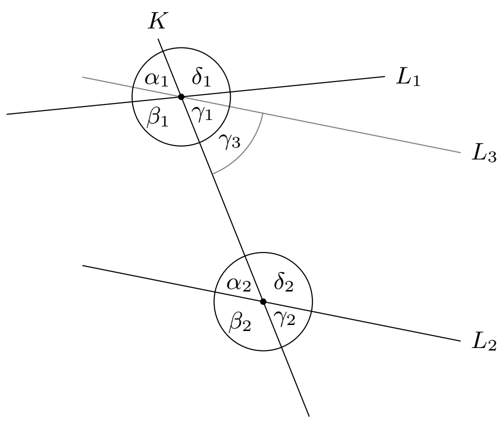

Figure 7: The construction in the proof of Theorem 2.23.

Corollary 2.24 *Let L_1 and L_2 be distinct lines and K a transversal. If* **one** *pair of alternate angles are equal, then L_1 and L_2 are parallel.*

Proof: If $\alpha_2 = \gamma_1$, then proceed as in the proof of Theorem 2.23. Suppose $\beta_1 = \delta_2$. Then, $\beta_1 + \gamma_1 = \pi$ and $\alpha_2 + \delta_2 = \pi$, rearranging and substituting into the original equation will yield the first case $\alpha_2 = \gamma_1$ anyway. □

Proposition 2.25 *Let L_1 and L_2 be distinct lines and K a transversal. If L_1 and L_2 are parallel, then the corresponding angles are equal.*

Proof: By Theorem 2.23, since L_1 and L_2 are parallel, we know that the alternate angles are equal. In particular, $\alpha_2 = \gamma_1$ and $\beta_1 = \delta_2$. By Proposition 2.21, we know that vertically-opposite angles are equal. In particular, $\alpha_1 = \gamma_1$ and $\beta_1 = \delta_1$. Combining these sets of equations gives that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. A near-identical argument works to show that $\gamma_1 = \gamma_2$ and $\delta_1 = \delta_2$. □

Note: The converse to Proposition 2.25 is also true. Combining this statement with its converse, we get this: corresponding angles are equal if and only if L_1 and L_2 are parallel.

Exercise 4 State and prove the converse to Proposition 2.25.

[**Hint:** Consider the angles $\alpha_1, \delta_1, \delta_2$ and combine Proposition 2.21 with Corollary 2.24.]

We can use some of the theory we have of angles to prove more important results.

Theorem 2.26 *The sum of the interior angles in a triangle is π.*

Proof: Let ABC be a triangle where $\angle CAB = \alpha, \angle ABC = \beta, \angle BCA = \gamma$. By the Parallel Postulate, we can draw the line L through B which is parallel to AC. We can see from Theorem 2.23 and Proposition 2.25 that the other angles between the (extended) line AB and L are as in Figure 8.

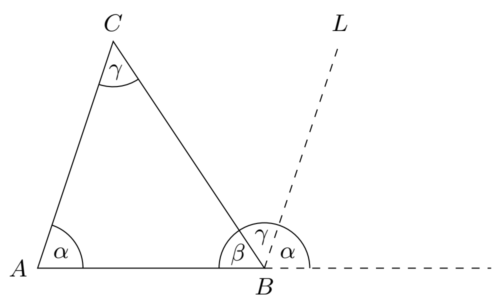

Figure 8: The setting for the proof of the sum of the interior angles of a triangle.

Indeed, the former result applies to angle γ and the latter result applies to angle α. Since angles on a straight line sum to π, we see precisely what we set out to prove: $\alpha + \beta + \gamma = \pi$. □

Definition 2.27 A bisector is a line which splits something exactly into two halves.

- An angle bisector of $\angle ABC$ is a ray from B dividing the angle into two equal angles.
- A perpendicular bisector of AB is a line perpendicular to AB through its midpoint.

Theorem 2.28 *Let L be a line and P be a point* **not** *on L. One can* drop a perpendicular *from P to L, i.e. there is a line segment PX, with X on L, which is perpendicular to L.*

Sketch of Proof: Given that L splits the plane into two sides, P will be on one of those sides. As such, let D be a point **not** on L but on the opposing side to P. We can construct a circle with centre P and radius PD by Axiom 2.3, specifically E3. By construction, L will intersect this circle at two points, called E and F. We can bisect the line segment EF, that is find its midpoint X. We then simply form line segments by connecting P to each of E, F, X. Then, the line segment PX is perpendicular to L, by construction. This is all pictured in Figure 9. □

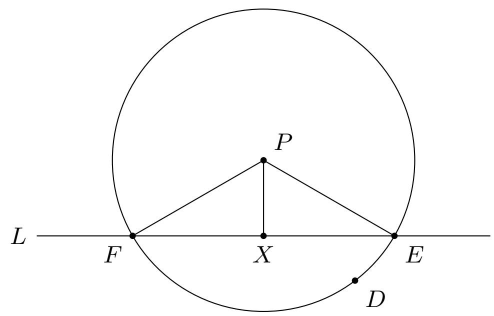

Figure 9: The foundations for the existence of a dropped perpendicular.

Definition 2.29 The distance from P to L is the length of the dropped perpendicular.

Remark 2.30 The above is only a sketch of a proof because we do not fully justify why this construction works. It might be a bit difficult if this is your first time looking at content like this but it is possible to flesh out the details a bit. On this one occasion, I will provide more (so you are comfortable with a proof vs a sketch of a proof). Indeed, we know that $PE = PF$ because both are radii of a circle, which is known to be constant. Since X is the midpoint of EF, we know that $EX = FX$. Therefore, by the SSS criterion, we know that $PXE \simeq PXF$. In particular, $\angle PXE = \angle PXF$. Since both PX and L are straight line (segment)s which meet at equal angles, said angles are both right angles, as required.

Definition 2.31 A parallelogram $ABCD$ has four vertices and edges AB, BC, CD, DA where each pair of opposite sides (AB, CD) and (AD, BC) is parallel.

Note: In general, we label a shape $ABCDE \cdots$ with vertices going around anticlockwise.

Throughout the discussion, we use height to mean *perpendicular* height, that is the height drawn from a chosen base edge which meets that edge at right angles.

Proposition 2.32 *Opposite sides and opposite angles in a parallelogram are equal.*

Proof: Let $ABCD$ be a parallelogram and consider its diagonal BD, as in Figure 10.

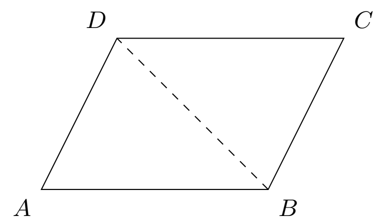

Figure 10: The parallelogram $ABCD$ with its diagonal BD.

By Definition 2.31, we have that AB and CD are parallel, with BD intersecting both (if we extend these segments to lines, we get a picture similar to Figure 6). By Theorem 2.23, we see that $\angle ABC = \angle BDC$. Similarly, we know that AD and BC are parallel and BD intersecting them both also, so the same theorem implies that $\angle ADB = \angle DBC$. Thus, the ASA condition applies, giving us $ABD \simeq BCD$. Thus is sufficient; the congruence guarantees that opposite sides are of the same length and angles of the same size (we could actually conclude the angles fact before applying ASA). □

Corollary 2.33 *The diagonal of a parallelogram bisects it.*

Proposition 2.34 *Parallelograms with the same base and height have equal areas.*

Proof: Let $ABCD$ and $ABEF$ be two parallelograms sharing the base AB, as in Figure 11.

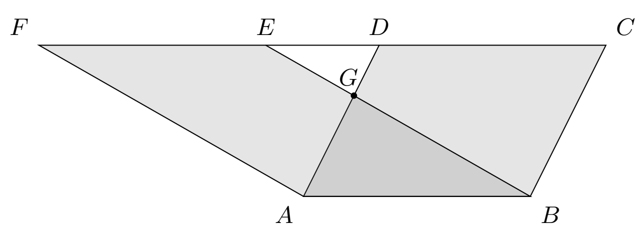

Figure 11: The parallelograms $ABCD$ and $ABEF$.

Thus, we have that $DC = AB = ED$ by Proposition 2.32. From this, it follows that $CE = DF$. The latter result also guarantees that $AD = BC$ and Proposition 2.25 implies $\angle CEB = \angle ACE$ as they are corresponding. By SAS, we get $ADF \simeq BCE$. In particular, they necessarily have the same area (this is actually hidden in the definition of congruence; all congruent shapes have equal areas – this is what gives area its meaning). If we subtract the area DEG, from each triangle, we get two four-sided shapes $AGEF$ and $BGDC$ of equal areas. Thus, adding the area ABG to each preserves the equality, but adding this gives us the areas for each parallelogram. □

Exercise 5 State and prove the triangle version of Proposition 2.34.

[**Hint:** Use Proposition 2.34; draw a picture and tweak it to make it look like Figure 11.]

Note: Both Proposition 2.34 and Exercise 5 can be generalised to two parallelograms (resp. triangles) with *equal* base lengths and heights will have the same area, not just two parallelograms (resp. triangles) with exactly the same bases. We won't go into detail on this as it isn't very illuminating; it uses the proof of Proposition 2.34 (resp. Exercise 5).

Corollary 2.35 *The area of a parallelogram is bh, for b the base length and h the height.*

Proposition 2.36 *A parallelogram with the same base and height as a triangle has twice the triangle's area.*

Proof: Suppose we have a parallelogram $ABCD$ and a triangle ABE, as in Figure 12.

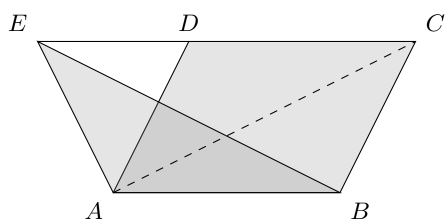

Figure 12: The parallelogram $ABCD$ and triangle ABE.

If we draw the diagonal AC of the parallelogram, then we can see that the areas of triangles ABE and ABC are equal by Exercise 5. But now, Corollary 2.33 tells us that AC bisects the parallelogram, so its area is twice that of the area of triangle ABE, as needed. □

Corollary 2.37 *The area of a triangle is $bh/2$, for b the base length and h the height.*

Definition 2.38 A triangle is a right-angled triangle if it contains a right angle.

Theorem 2.39 (Pythagoras' Theorem) *Let ABC be a triangle with side lengths a, b, c opposite vertices A, B, C respectively. Then, $\angle BCA = \pi/2$ if and only if $a^2 + b^2 = c^2$.*

Proof: ($\Rightarrow$) First suppose that the triangle is right-angled, with the angle at vertex C. Consider the square with side lengths $a + b$ formed from triangles congruent to ABC, as in Figure 13.

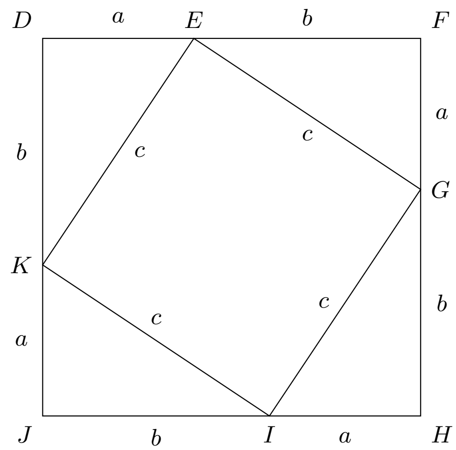

Figure 13: The square in the proof of Pythagoras' Theorem.

As noted, the four right-angled triangles are congruent to ABC, hence the labelling in Figure 13. Recall that Theorem 2.26 guarantees that the sum of the interior angles of a triangle is π; it follows from this that $\angle DEK + \angle FEG = \pi/2$, but then angles on a straight line sum to π, so $\angle KEG = \pi/2$. Similar reasoning works also for the vertices G, I, K. The conclusion: we indeed see that $EGIK$ is a square. Now, we can use Corollary 2.37 to write the area of the square $DFHJ$ in two different-yet-equal ways:

$$(a + b)^2 = 4(ab/2) + c^2 \quad \Leftrightarrow \quad a^2 + 2ab + b^2 = 2ab + c^2,$$

where the left-hand-side is simply the area of the square $DFHJ$ and the right-hand-side is the sum of the areas of the triangles and the inner square $EGIK$. If we expand this out and rearrange, we immediately conclude that $a^2 + b^2 = c^2$.

($\Leftarrow$) Suppose now that ABC is a triangle such that $a^2 + b^2 = c^2$. We will construct a new shape. Draw two line segments $B'C'$ and $A'C'$ of lengths a and b, respectively, such that they are perpendicular. Then, we can complete this to a triangle $A'B'C'$ by adding the line segment $A'B'$ of length c', say. By construction, this is a right-angled triangle with side lengths a, b, c' opposite vertices A', B', C' respectively. Therefore, the forward direction of Pythagoras' Theorem (proved above) applies, meaning that $a^2 + b^2 = (c')^2$. Combining this with what we assumed at the beginning, we have $c = c'$. Hence, SSS congruence tells us that $A'B'C' \simeq ABC$. In particular, ABC is a right-angled triangle. □

Exercise 6 A trapezium is a four-sided shape with one pair of parallel edges, one such example is the shape $DEFG$ in Figure 14, with parallel sides DE and FG. The area of a trapezium is computed as follows:

(i) Add the lengths of the two parallel sides.

(ii) Multiply this by the (perpendicular) height.

(iii) Divide the result by two.

Now you are equipped with this, give an alternate proof of Pythagoras' Theorem (the forward direction) by computing the area of Figure 14, as we did for Figure 13 above.

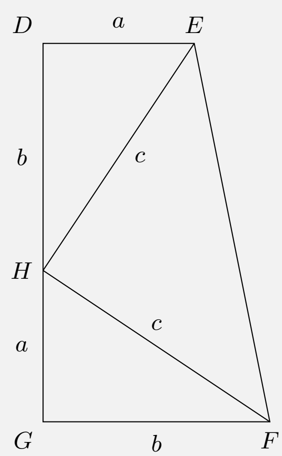

Figure 14: The trapezium for an alternate proof of Pythagoras' Theorem.

Corollary 2.40 *Let ABC be a right-angled triangle with side lengths a, b, c opposite vertices A, B, C respectively and $\angle BCA = \pi/2$. Then, $a + b > c$.*

Proof: Assume to the contrary that $a + b \leq c$. Then, squaring both sides gives us $(a + b)^2 \leq c^2$, which expands out to $a^2 + 2ab + b^2 \leq c^2$. However, Pythagoras' Theorem tells us that $a^2 + b^2 = c^2$, so we can re-write this inequality as $c^2 + 2ab \leq c^2$, which is obviously equivalent to $2ab \leq 0$, which is a contradiction since $a > 0$ and $b > 0$ (because they are side lengths). □

Let ABC be a right-angled triangle with side lengths a, b, c opposite vertices A, B, C respectively and $\angle BCA = \pi/2$. We denote the angle $\angle ABC = \theta$. Then, we can 'define' the following:

- The *sine* of angle θ is $\sin(\theta) = b/c$.
- The *cosine* of angle θ is $\cos(\theta) = a/c$.

- The *tangent* of angle θ is $\tan(\theta) = b/a$.

Note: This is a very generous use of the word 'define' because we will see a much more rigorous definition of the trigonometric functions in Chapter **??**. As such, we have done away with the usual pomp of putting it in a blue box and assigning it a definition number.

Remark 2.41 This adheres to the informal rule taught to high school students:

- The sine is found by doing the opposite ÷ hypotenuse.
- The cosine is found by doing the adjacent ÷ hypotenuse.
- The tangent is found by doing the opposite ÷ adjacent.

Recall that the hypotenuse is defined as the longest edge in a right-angled triangle, that is the edge opposite the right angle. At least here in the UK, we remember this as 'SOHCAHTOA', which stands for $S = O/H$ and $C = A/H$ and $T = O/A$. It is encouraged to sketch a picture of triangle ABC with the labels defined above to visualise $\sin(\theta), \cos(\theta), \tan(\theta)$.

Proposition 2.42 *For any angle θ, we have $\sin(\theta)/\cos(\theta) = \tan(\theta)$.*

Sketch of Proof: This follows from the informal discussion in Remark 2.41. □

Theorem 2.43 (Cosine Rule) *Let ABC be a triangle with side lengths a, b, c opposite vertices A, B, C respectively and $\angle BCA = \alpha$. Then, $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$.*

Proof: We do the proof assuming all angles in the triangle are acute, that is strictly less than $\pi/2$ but the result holds in other cases. We will drop a perpendicular from C to the edge AB to the point P. We will say h is the length of this line segment CP. Then, we have two right-angled triangles ACP and BCP on which to apply Pythagoras' Theorem. Before proceeding, we will create Figure 15 to assist in our visualisation.

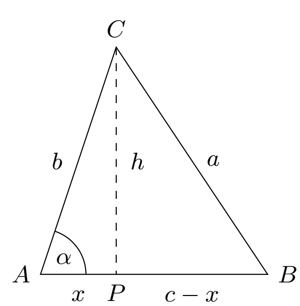

Figure 15: The triangle in the proof of the Cosine Rule.

If we say that AP has length x, it follows that BP has length $c - x$ (since the whole segment AB has length c). We can truly now apply Pythagoras' Theorem on each triangle:

$$x^2 + h^2 = b^2 \qquad \text{and} \qquad (c - x)^2 + h^2 = a^2$$

Subtracting one from the other, we see that

$$\begin{aligned} a^2 - b^2 &= (c - x)^2 - x^2 \\ &= c^2 - 2cx + x^2 - x^2 \\ &= c^2 - 2cx. \end{aligned}$$

The final thing to note is that $\cos(\alpha) = x/b$ which is equivalent to $x = b\cos(\alpha)$. If we substitute this in and add b^2 to both sides of the above, we get the result. □

Exercise 7 Determine what happens in the degenerate case $\alpha = \pi/2$.

[**Hint:** You can work out the value of $\cos(\pi/2)$ from the identity $\cos(\pi - \theta) = -\cos(\theta)$.]

3 Coordinate Two-Dimensional Geometry

An alternative to the axiomatic geometric approach discussed so far is to introduce coordinates. The most standard form of coordinates is Cartesian coordinates, named after René Déscartes who played an important role in the development of this discipline.

Definition 3.1 A coordinate system is a choice of the following:

- An origin, O.
- A pair of perpendicular axes that intersect at O.
- A scale on each axis.
- An orientation.

In words, Definition 3.1 says that we can form coordinates in two-dimensions by specifying a reference point (the origin) and a notion of how far away from said origin we move (scaled axes), where we write this movement as a pair of numbers, one each for how far we move parallel to the axes (orientation).

The most standard coordinate system is Cartesian coordinates, with the usual x-axis and y-axis going through the origin $O = (0, 0)$. The scale allows for any real number to appear on each axis. Therefore, we can think of our two-dimensional plane as the set of pairs of real numbers, that is $\mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$. Recall this is the Cartesian product from Definition **??**, hence the name of the coordinate system.

Note: For shorthand, we write $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ to mean the two-dimensional plane. This notation sets us up for discussing three-dimensional space in Section 5, but even more so it establishes that we can define higher-dimensional spaces (e.g. $\mathbb{R}^n$ is n-dimensional).

Definition 3.2 An equation of a line is $ax + by + c = 0$, for $a, b, c \in \mathbb{R}$ not all zero.

Remark 3.3 You may be familiar with the equation $y = mx+p$ of a line, where m is the 'gradient' of the line and p is the 'y-intercept' of the line. This is almost good, but this expression will never work for vertical lines of the form $x = k$, for any $k \in \mathbb{R}$. Moreover, any line in $\mathbb{R}^2$ can be expressed as $ax + by + c = 0$ for an appropriate choice of a, b, c but the converse is **not** true:

- If we choose $a = 0, b = 0, c = 0$, the equation describes the whole plane $\mathbb{R}^2$.

- If we choose $a = 0, b = 0, c \neq 0$, the equation describes the empty set $\emptyset$.

Exercise 8 Demonstrate that the equation $ax + by + c = 0$ can describe both our usual line $y = mx + p$ and our vertical line $x = k$. In other words, choose values for a, b, c so that the equation rearranges to get $y = mx + p$ (and again so that it rearranges to $x = k$).

Our goal is to now better understand this Cartesian coordinate description of lines. In particular, we wish to know what the least amount of information is needed to determine a line and how, with that information, we can find the equation of the line.

Definition 3.4 The gradient of a non-vertical line L that passes through distinct (x_1, y_1) and (x_2, y_2) is the number $m = (y_2 - y_1)/(x_2 - x_1)$.

Lemma 3.5 *Let L be a line with gradient m such that it meets the x-axis at angle θ, measured anti-clockwise from the x-axis. Then,* $\tan(\theta) = m$.

Proof: (i) If $\theta = 0$, L is the x-axis and has gradient 0; we recover the known fact $\tan(0) = 0$.

(ii) If $\theta \in (0, \pi/2)$ is acute, we let $(x_0, 0)$ be the x-intercept of L and we choose another point (x_1, y_1) on L such that $x_1 > x_0$. Dropping a perpendicular from (x_1, y_1) to the x-axis, i.e. to the point $(x_1, 0)$, we obtain a right-angled triangle with vertices $(x_0, 0), (x_1, 0), (x_1, y_1)$. By the discussion in and around Remark 2.41, we see that $\tan(\theta) = y_1/(x_1 - x_0)$, which is precisely m.

(iii) If $\theta \in (\pi/2, \pi)$ is obtuse, we can look to the other angle φ formed between the x-axis and L (which will be acute). Proceeding as above but with φ, we see that $\tan(\varphi) = -m$ and, consequently, it follows that $\tan(\theta) = \tan(\pi - \varphi) = -\tan(\varphi) = m$. $\square$

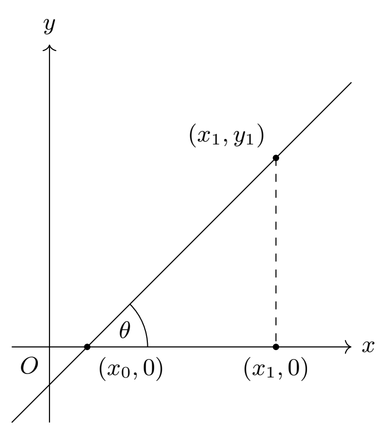

Figure 16: The line L in part (ii) of the proof of Lemma 3.5.

Note: We take some liberties in the proof of Theorem 3.6 below by using so-called 'compound-angle formulae'. These will be formally introduced later (you may have seen some of them in your mathematical youth) when we properly discuss the trigonometric functions. For now, we note the specific ones used in the next proof:

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y),\\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y),\\
\tan(x+y) &= \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}.
\end{aligned}$$

Theorem 3.6 *Let L_1 and L_2 be lines with gradients $m_1 \neq 0$ and m_2, respectively. Then, L_1 and L_2 are perpendicular if and only if $m_2 = -1/m_1$.*

Proof: ($\Rightarrow$) Let θ be the angle between L_1 and the x-axis, again measured anti-clockwise from the x-axis. Assume first that $\theta \in (0, \pi/2)$ is acute. Because L_1 and L_2 are assumed perpendicular, this means that rotating L_1 by $\pi/2$ about their point of intersection will transform it onto L_2. Now, we can apply Lemma 3.5 to these two lines:

$$\begin{aligned}
m_2 &= \tan\big(\theta + \pi/2\big)\\
&= \frac{\sin\big(\theta+\pi/2\big)}{\cos\big(\theta+\pi/2\big)}\\
&= \frac{\sin(\theta)\cos\big(\pi/2\big) + \cos(\theta)\sin\big(\pi/2\big)}{\cos(\theta)\cos\big(\pi/2\big) - \sin(\theta)\sin\big(\pi/2\big)}\\
&= \frac{\cos(\theta)}{-\sin(\theta)}\\
&= -\frac{1}{\tan(\theta)}\\
&= -\frac{1}{m_1}.
\end{aligned}$$

A near-identical argument works for when $\theta \in (\pi/2, \pi)$ is obtuse.

($\Leftarrow$) Suppose that $m_2 = -1/m_1$ and let both $m_1 = \tan(\theta)$ and $-1/m_1 = \tan(\varphi)$. By assumption, we have $-1/\tan(\theta) = \tan(\varphi)$ which rearranges to $1 - \tan(\theta)\tan(\varphi) = 0$. Using a compound-angle formula, this implies that $\tan(\theta + \varphi)$ is undefined, but this can only be the case when $\theta + \varphi = \pi/2$ (or an odd-multiple of it, but our angles are restricted to $(0, \pi)$ only). Hence, the angle between L_1 and L_2 is a right angle, as required. □

Theorem 3.7 *Consider an arbitrary line L.*

(i) *If L is non-vertical, passes through (x_1, y_1) and has gradient m, then it is given by*

$$y - y_1 = m(x - x_1).$$

(ii) *If L is non-vertical and passes through distinct $(x_1, y_1), (x_2, y_2)$, then it is given by*

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \textit{for all } x \neq x_1.$$

(iii) *If L passes through $(p, 0)$ and $(0, q)$ for $p, q \neq 0$, then it is given by*

$$\frac{x}{p} + \frac{y}{q} = 1.$$

Proof: (i) Using the $y = mx + p$ form, since L is assumed non-vertical, it can be described by $y_1 = mx_1 + p$, implying that the y-intercept $p = y_1 - mx_1$. Therefore, the equation of the line is $y = mx + y_1 - mx_1$; this easily rearranges to the desired formula.

(ii) Since L is non-vertical, it satisfies (i), i.e. it is described by the equation $y - y_1 = m(x - x_1)$. Therefore, assuming that $x \neq x_1$, we can divide this to get the following expression:

$$m = \frac{y - y_1}{x - x_1}.$$

Alternatively, Definition 3.4 gives us the gradient in terms of the two distinct points (x_1, y_1) and (x_2, y_2), so equating this with our expression for m above gives the result.

(iii) Consider the general equation of a line $ax + by + c = 0$. We can substitute $(x, y) = (p, 0)$ and $(x, y) = (0, q)$ in turn into this equation. Doing so yields

$$ap + c = 0 \quad \text{and} \quad bq + c = 0.$$

We assumed that $p, q \neq 0$, so we can rearrange these to get $a = -c/p$ and $b = -c/q$. Hence, the equation of our line is $-cx/p - cy/q + c = 0$. We can factorise this to $-c(x/p + y/q - 1) = 0$. Because $c \neq 0$ (if it was, all three of $a, b, c = 0$ which is not allowed in Definition 3.2), this final equation holds precisely when $x/p + y/q = 1$. □

What are the benefits of this coordinate description of the two-dimensional plane? Perhaps things look a little more concrete than the justifications involving seemingly-arbitrary axioms. One useful benefit is that we get a neat formula for the distance described in Definition 2.29. First, we provide an exercise which informs the next result.

Exercise 9 Prove the distance between two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is given by

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

[**Hint:** This is an application of Pythagoras' Theorem; draw a picture.]

Proposition 3.8 *Let L be a line given by $ax + by + c = 0$ with $a, b \neq 0$ and $P = (x_1, y_1)$ be a point not on L. Then, the distance between P and L is*

$$\operatorname{dist}(P, L) = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Proof: Rearranging the equation of L shows that it has gradient $-a/b$. Hence, Theorem 3.6 implies that the perpendicular dropped from P to L will have gradient b/a. By Theorem 3.7(i), the perpendicular line $L^\perp$ is described by $y - y_1 = \frac{b}{a}(x - x_1)$. Written in standard form, this becomes $bx - ay + ay_1 - bx_1 = 0$. To find the base of the dropped perpendicular, that is the intersection of L with $L^\perp$, it amounts to solving the equations of each line simultaneously. Indeed, this gives us the intersection point

$$(x_2, y_2) = \left(\frac{b^2x_1 - aby_1 - ac}{a^2 + b^2}, \frac{a^2y_1 - abx_1 - bc}{a^2 + b^2} \right).$$

Using Exercise 9, we know that $\operatorname{dist}(P, L) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and so substituting the expressions for x_2 and y_2 above will give the result. □

Note: The notation $L^\perp$ refers to a line perpendicular to L. The symbol $\perp$ is actually called the 'perp' symbol. This will be used later when we look at vectors and describing orthogonality (which is another word for perpendicularity).

Example 3.9 Consider the line L given by $y = 3x - 4$ and the point $P = (3, 2)$. It is first clear that P is not on L, since $3(3) - 4 = 5 \neq 2$. Second, the standard form of the line is $-3x + y + 4 = 0$ so we certainly see that $a, b \neq 0$. Hence, we are in a position to use Proposition 3.8. Indeed,

$$\operatorname{dist}(P, L) = \frac{\left|-3(3) + 1(2) + 4\right|}{\sqrt{(-3)^2 + 1^2}} = \frac{|-3|}{\sqrt{10}} = \frac{3}{\sqrt{10}}.$$

Note: The absolute value $|x| = \sqrt{x^2}$ for any x; this is one way to generalise Definition **??** and we will use it often. An equivalent generalisation is given in Definition **??**.

We have spent time looking at the coordinate form of a line. In Section 4, we will look at some more interesting curves. Before that, there is another fundamental geometric object that we should be able to describe: a circle. Here, we will see a more rigorous (but still not the best) introduction to the trigonometric functions.

Definition 3.10 Let $\theta \in \mathbb{R}$ and, starting at the point $(1,0) \in \mathbb{R}^2$, let P be the point achieved by rotating $(1,0)$ by angle θ about the origin. The convention is that the rotation is anti-clockwise for $\theta > 0$ and clockwise for $\theta < 0$.

- The cosine of θ is defined as the x-coordinate of P.
- The sine of θ is defined as the y-coordinate of P.

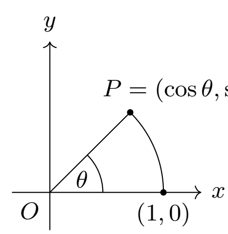

Figure 17: The descriptions of $\cos(\theta)$ and $\sin(\theta)$ from Definition 3.10.

Note: We will still refer to cosine and sine as trigonometric *functions*, despite the fact it is not clear from any description of them given so far that they are functions. This will be clarified when we give them their the most rigorous treatment in terms of power series.

Lemma 3.11 *The cosine and sine definitions in* Remark 2.41 *and* Definition 3.10 *coincide.*

Proof: Let P be the point achieved by rotating $(1,0)$ about the origin by θ, as in Figure 17. If we drop a perpendicular from P to the x-axis, i.e. to the point $X = (\cos\theta, 0)$, then we see that OPX is a right-angled triangle where $\angle POX = \theta$. There are three cases to consider. Throughout, $\cos_{\text{old}}$ and $\sin_{\text{old}}$ denote the older definitions presented in Remark 2.41 whereas cos and sin denote the better-introduced newer versions in Definition 3.10.

(i) If $\theta \in (0, \pi/2)$ is acute, then we see that

$$\cos_{\text{old}}(\theta) = OX/OP = \cos(\theta), \qquad \sin_{\text{old}}(\theta) = PX/OP = \sin(\theta).$$

(ii) If $\theta = \pi/2$, then we see that

$$\cos_{\text{old}}(\theta) = 0 = \cos(\theta), \qquad \sin_{\text{old}}(\theta) = 1 = \sin(\theta).$$

(iii) If $\theta \in (\pi/2, \pi)$ is obtuse, then $P = (-\cos\varphi, \sin\varphi)$ where $\varphi = \pi - \theta$ and we see that

$$\cos_{\text{old}}(\theta) = -\cos(\varphi) = \cos(\theta), \qquad \sin_{\text{old}}(\theta) = \sin(\varphi) = \sin(\theta).$$ □

As we can see from Figure 17, there may be a relationship between these trigonometric functions and circles, since a circle is just a rotation by a fixed length (the radius) about a point. Before moving over to describing said shapes, we will first note a number of useful properties of these trigonometric functions.

Proposition 3.12 *Let $\theta \in \mathbb{R}$ be treated as a signed angle.*

(i) $\sin(\theta + 2\pi) = \sin(\theta)$.
(ii) $\cos(\theta + 2\pi) = \cos(\theta)$.
(iii) $\sin(\pi - \theta) = \sin(\theta)$.
(iv) $\cos(\pi - \theta) = -\cos(\theta)$.
(v) $\sin(-\theta) = -\sin(\theta)$.
(vi) $\cos(-\theta) = \cos(\theta)$.
(vii) $\sin^2(\theta) + \cos^2(\theta) = 1$.
(viii) $\sin\big(\theta + \pi/2\big) = \cos(\theta)$.

Exercise 10 Prove (or justify some of) Proposition 3.12.

[**Hint:** (iii) and (iv) follow from the *old* definitions; (v) and (vi) follow from Figure 17; (vii) is Pythagoras' Theorem; (viii) is simply a compound-angle formula.]

Even though we have (likely) all seen circles before, and even referenced them earlier in these notes, we now provide a definition for what a circle is, before we proceed in prescribing a coordinate description.

Definition 3.13 A circle is the set of points a fixed distance away from a chosen point. We call the fixed distance the radius and the chosen point the centre.

Remark 3.14 A tempting alternate definition of a circle may be this:

A circle is a shape with constant width, where we call the width the diameter.

This is actually **not** sufficient to describe only a circle. Indeed, there are plenty of other shapes that have a constant width which are not circles. Real-world examples include 20p and 50p coins in the UK. The width is defined as the distance between parallel supporting lines, that is the pair of parallel lines on either side of the shape that each touch the shape at precisely one point. We can see an example of such a shape in Figure 18 below.

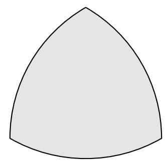

Figure 18: A shape of constant width known as a Reuleaux triangle.

Definition 3.15 The equation of a circle (or locus equation of a circle) with radius $r > 0$ and centre (a, b) is $(x-a)^2 + (y-b)^2 = r^2$.

Note: This is a direct consequence of Pythagoras' Theorem, specifically of Exercise 9.

Proposition 3.16 (Parametric Equation of a Circle) *A point $P = (x, y)$ lies on the circle of radius r and centre (a, b) if and only if there is a $t \in [0, 2\pi)$ with $P = (a+r\cos t, b+r\sin t)$.*

Proof: ($\Leftarrow$) If said t exists, then $(x-a)^2+(y-b)^2 = (r\cos t)^2+(r\sin t)^2 = r^2(\cos^2 t+\sin^2 t) = r^2$, where we have used Proposition 3.12(vii) to get the final equality.

($\Rightarrow$) Suppose P lies on the circle of radius r and centre $A = (a, b)$. Define the point $Q = (a+r, b)$. Then, the line segment AQ is a horizontal radius of the circle, and the segment AP is obtained from AQ via a rotation of angle $t \in [0, 2\pi)$ about the centre A. In the case that $t \in (0, \pi/2)$ is acute, it is clear that

$$\cos(t) = \frac{x-a}{r} \quad \text{and} \quad \sin(t) = \frac{y-b}{r},$$

from which they rearrange to give $x = a + r\cos(t)$ and $y = b + r\sin(t)$, as required. One can proceed similarly for the other values of t. $\square$

Remark 3.17 It is immensely useful to have parametric equations for some of the curves that we see in geometry. Why? Because all one needs to do to generate points on such a curve is substitute in a number. On the other hand, with our usual description of a line or circle, say, we need to substitute in one coordinate (either x or y) and rearrange for the other; this is much more work. We will see much later on in Chapter **??** how parametrising curves (and surfaces) can make our mathematical lives easier.

For the end of this section, we move away from the Cartesian coordinate system (although this is the system we will use almost always) to discuss an alternative, and very useful, set of coordinates which will allow us to describe curves in a neat way. Ultimately, this will inform the discussion in Section 4.

Definition 3.18 The system of polar coordinates is defined by the following:

- A pole, O.
- A polar axis X through the pole.
- A scale on the polar axis.

In this way, any point P in the plane is specified by the distance r from the pole and the angle θ between the line segment PO and X. Polar coordinates are denoted (r, θ).

Note: Again, we use the convention that $\theta > 0$ is an anti-clockwise rotation and $\theta < 0$ is a clockwise rotation. Generally, we also restrict either to $\theta \in [0, 2\pi)$ or to $\theta \in (-\pi, \pi]$.

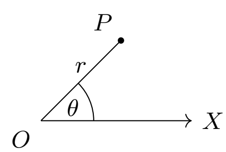

Figure 19: The plane as described by polar coordinates.

Lemma 3.19 *Let P be a point in the plane with polar coordinates (r, θ). Then, it has Cartesian coordinates $(x, y) = (r\cos\theta, r\sin\theta)$.*

Proof: By Definition 3.10, the point arrived at by rotating $(1, 0)$ about O by angle θ is precisely $(\cos\theta, \sin\theta)$. Since P is obtained by rotating $(r, 0)$ around O by angle θ, it is an easy consequence that it has Cartesian coordinates $(r\cos\theta, r\sin\theta)$. □

Example 3.20 Suppose we wish to convert the polar point $(\sqrt{2}, 5\pi/4)$ to Cartesian coordinates. Then, Lemma 3.19 tells us that $x = \sqrt{2}\cos\big(5\pi/4\big) = -1$ and $y = \sqrt{2}\sin\big(5\pi/4\big) = -1$.

Note: In general, for $a \in \mathbb{R}$, the number $\tan^{-1}(a)$ isn't well-defined because if $\tan(\theta) = a$, then $\tan(\theta + n\pi) = a$ for all $n \in \mathbb{Z}$. Thus, by $\tan^{-1}(a)$, we restrict to $\theta \in (-\pi/2, \pi/2)$.

Lemma 3.21 *Let P be a point in the plane with Cartesian coordinates (x, y). Then, it has polar coordinates (r, θ), where*

$$r = \sqrt{x^2 + y^2} \quad \textit{and} \quad \theta = \begin{cases} \tan^{-1}(y/x), & \textit{if } x > 0 \\ \tan^{-1}(y/x) + \pi, & \textit{if } x < 0 \\ \pi/2, & \textit{if } x = 0 \textit{ and } y > 0 \\ -\pi/2, & \textit{if } x = 0 \textit{ and } y < 0 \\ \textit{undefined}, & \textit{if } x = 0 \textit{ and } y = 0 \end{cases}.$$

Proof: There are a number of cases to consider.

(i) If P is in quadrant one $(x > 0, y > 0)$, then $\theta \in (0, \pi/2)$ which is simply $\tan^{-1}(y/x)$.

(ii) If P is in quadrant two $(x < 0, y > 0)$, then $\theta \in (\pi/2, \pi)$, so there is an acute angle $\alpha = \pi - \theta$. Substitute in $\alpha = \tan^{-1}(y/-x) = -\tan^{-1}(y/x)$ and rearrange.

(iii) If P is in quadrant three $(x < 0, y < 0)$, then $\theta \in (\pi, 3\pi/2)$, so there is an acute angle $\alpha = \theta - \pi$. Substitute in $\alpha = \tan^{-1}(-y/-x) = \tan^{-1}(y/x)$ and rearrange.

(iv) If P is in quadrant four $(x > 0, y < 0)$, then $\theta \in (3\pi/2, 2\pi)$, so there is an acute angle $\alpha = 2\pi - \theta$. Substitute in $\alpha = \tan^{-1}(-y/x) = -\tan^{-1}(y/x)$ and rearrange.

Now, $\tan^{-1}(y/x)$ is undefined when $x = 0$, meaning $\theta = \pm\pi/2$ depending on the orientation. Finally, the fact that $r = \sqrt{x^2 + y^2}$ is an easy application of Pythagoras' Theorem. □

Example 3.22 Suppose we wish to convert the Cartesian point $(-1, 1)$ to polar coordinates. Then, Lemma 3.21 tells us that $r = \sqrt{2}$ and that the angle $\theta = \tan^{-1}(-1) + \pi = 3\pi/4$.

Exercise 11 Express the polar point $(2, \pi)$ in Cartesian coordinates. Furthermore, express the Cartesian point $(0, 0)$ in polar coordinates. Is your answer unique?

Proposition 3.23 *Let (x, y) be Cartesian coordinates. Then, the coordinate system (X, Y) obtained by rotating the (x, y)-axes through angle α about the origin is described by*

$$X = x\cos(\alpha) + y\sin(\alpha),$$
$$Y = -x\sin(\alpha) + y\cos(\alpha).$$

Proof: Let P be a point in the plane with Cartesian coordinates (x, y) and polar coordinates

(r, θ). Suppose that (R, Θ) are the 'new' polar coordinates of P, that is where we take the polar axis to be the X-axis. Then, $R = r$ and $\Theta = \theta - \alpha$. With this information, we can use some compound-angle formulae with Lemma 3.19 to get the desired result:

$$\begin{aligned} X &= R\cos(\Theta) \\ &= r\cos(\theta - \alpha) \\ &= r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\ &= x\cos(\alpha) + y\sin(\alpha) \end{aligned}$$

and

$$\begin{aligned} Y &= R\sin(\Theta) \\ &= r\sin(\theta - \alpha) \\ &= r(\sin\theta\cos\alpha - \cos\theta\sin\alpha) \\ &= y\cos(\alpha) - x\sin(\alpha). \end{aligned}$$

□

4 Conic Sections

Conic sections (or conics, for short) have been studied for thousands of years. There are second-hand accounts pertaining to the study of these curves around the year 320BC. Sadly, neither the work nor the names for the conics survived. One of the most natural ways to view a conic is to take a cone and intersect it with a plane; the boundaries of the resulting sets will be what we call conic sections. This will be illustrated but another, more algebraic definition is now given.

Definition 4.1 A non-degenerate conic section (or conic) is is the locus of a point in the plane whose distance from a fixed point is some positive constant multiple of its distance from a fixed line not containing the point.

- The fixed point is called the focus, F.
- The fixed line is called the directrix, L.
- The positive constant multiple is called the eccentricity, e.

Note: In other words, a point P lies on a conic if and only if $\text{dist}(P, F) = e\,\text{dist}(P, L)$.

Remark 4.2 As mentioned above, one beautiful interpretation of conics comes from considering the intersection of a cone with various planes. Here, a cone is a subset of $\mathbb{R}^3$ so lives in three-dimensional space and a plane is essentially just a copy of $\mathbb{R}^2$.

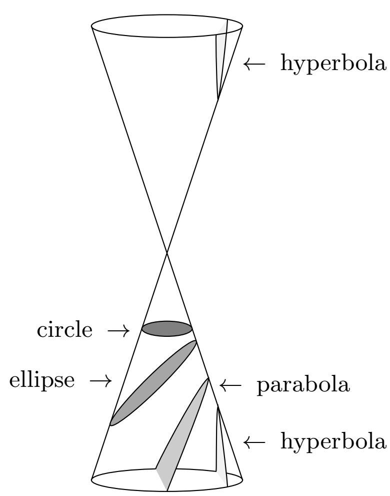

Figure 20: The conics arising from the intersection of a cone with various planes.

Here is a quick explanation of the orientation of the planes relative to the cone drawn in Figure 20: the circle is contained on a horizontal plane; the ellipse is contained on an angled plane through both sides of the cone; the parabola is contained on an angled plane through one side and the base of the cone; the hyperbola is contained on a vertical plane. We can see that the hyperbola is the only conic with more than 'one piece'.

The goal is to describe these conics using Cartesian coordinates. We will do this by considering different values of the eccentricity and studying the resulting conics. The values are as follows:

$$0 < e < 1, \qquad e = 1, \qquad e > 1.$$

Conic I: The Parabola

Definition 4.3 A parabola is the conic obtained when the eccentricity $e = 1$.

The Cartesian coordinate system will give us a neat interpretation of a parabola. Indeed, suppose the focus F lies on the x-axis, meaning it is of the form $F = (a, 0)$. Then, the directrix L is defined by $x = -a$. Using the previous note along with our formula for distance in Cartesian coordinates, we can see that

$$\begin{aligned} \text{dist}(P, F) = \text{dist}(P, L) &\Leftrightarrow \sqrt{(x-a)^2 + y^2} = |x + a| \\ &\Leftrightarrow x^2 - 2ax + a^2 + y^2 = (x+a)^2 \\ &\Leftrightarrow y^2 = 4ax. \end{aligned}$$

Immediately, this gives us an equation that any point on a parabola must satisfy.

Corollary 4.4 *The* standard form of a parabola *is* $y^2 = 4ax$ *for all* $a \neq 0$*.*

- *Its vertex is the point* $(0, 0)$*.*
- *Its focus is the point* $(a, 0)$*.*
- *Its directrix is the line* $x = -a$*.*

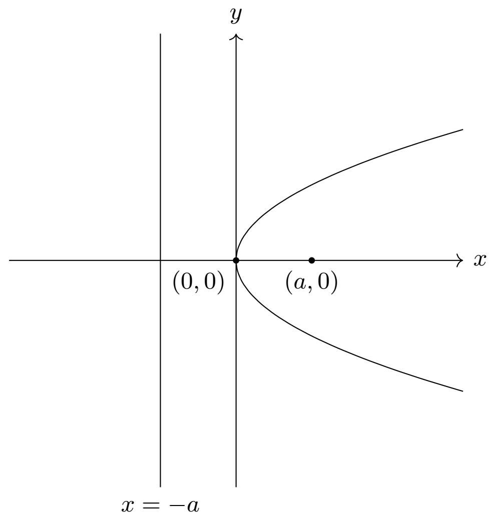

Figure 21: The standard parabola $y^2 = 4ax$ with its directrix.

Proposition 4.5 (Parametric Equation of a Parabola) *Let $a \neq 0$. A point $P = (x, y)$ lies on the parabola $y^2 = 4ax$ if and only if there exists $t \in \mathbb{R}$ such that $P = (at^2, 2at)$.*

Proof: ($\Rightarrow$) Given $y^2 = 4ax$, take $t = y/2a$. Hence, $y = 2at$ and $x = y^2/4a = 4a^2t^2/4a = at^2$.

($\Leftarrow$) Given $(x, y) = (at^2, 2at)$ for some $t \in \mathbb{R}$, we have $y^2 = 4a^2t^2 = 4a(at^2) = 4ax$, as needed. □

Example 4.6 Suppose we wish to sketch the parabola described by $y^2 - 6y + 3x = 10$. The goal is to transform this into the standard form. In fact, we do something more general but this will be discussed after this example. Indeed, we need to re-write this equation so that it has the form $Y^2 = 4aX$, for relevant coordinates (X, Y) in terms of (x, y). Essentially, the standard form has squared y-terms on the left and a multiple of the x-terms on the right. If we do this for the given equation, we obtain the following:

$$y^2 - 6y = -3x + 10 \quad \Leftrightarrow \quad (y-3)^2 = -3(x - 19/3),$$

where we complete the square and tidy things up. If we introduce the labels $x - 19/3 = X$ and $y - 3 = Y$, then this equation becomes

$$Y^2 = -3X \quad \Leftrightarrow \quad Y^2 = 4(-3/4)X,$$

so the standard form is such that $a = -3/4$. We know the features of a parabola (e.g. vertex, focus, directrix) when it is in the standard form, so we need only use our 'transformation' $X = x - 19/3$ and $Y = y - 3$ to find out what these features are for this particular parabola.

- Its vertex is $(X, Y) = (0, 0)$, which is to say $(x, y) = (19/3, 3)$.
- Its focus is $(X, Y) = (a, 0)$, which is to say $(x, y) = (67/12, 3)$.
- Its directrix is $X = -a$, which is to say $x = 85/12$.

Exercise 12 Actually sketch the parabola discussed in Example 4.6.

[**Hint:** Although we didn't do so in the above example, you should also label on where the parabola intercepts the y-axis, that is substitute in $x = 0$ and solve.]

Note: What we did by writing (X, Y) in terms of (x, y) is a coordinate transformation.

Remark 4.7 In Example 4.6, in order to sketch our parabola, we need to write it in standard form using a new coordinate system (X, Y) which is somehow related to our usual Cartesian coordinates (x, y). In fact, we wrote explicitly what the relation was: $X = x - 19/3$ and $Y = y - 3$. In essence, what we have done here is shifted our coordinate system so that the origin $(0, 0) \mapsto (19/3, 3)$, which allows us to easily obtain the standard form of the parabola. In general, if we move the origin $(0, 0) \mapsto (c, d)$ to a new point, then the new coordinates (X, Y) are related to the old coordinates (x, y) by the following formulae:

$$X = x + c,$$
$$Y = y + d.$$

This is captured in Figure 22 below, which shows how the Cartesian coordinate axes move to give us a new system of coordinates.

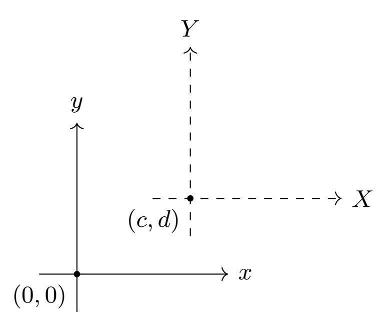

Figure 22: Transforming coordinates from (x, y) to (X, Y).

Conic II: The Ellipse

Definition 4.8 An ellipse is the conic obtained when the eccentricity $0 < e < 1$.

Deriving the standard form of an ellipse is not as easy as for the parabola case. However, we can introduce a bit of machinery which will provide us with an easier time; the next result will tell us what happens to the focus, directrix and conic under a general change of variables from (x, y) to (X, Y).

Proposition 4.9 *Consider a conic with eccentricity $e \neq 1$. It is possible to make a change of variables from (x, y)-coordinates to (X, Y)-coordinates such that the focus F becomes the origin and the directrix L is parallel to the Y-axis, that is of the form $X = -h$ for some $h \in \mathbb{R}$. Then, the conic itself satisfies*

$$\left(X - \frac{he^2}{1-e^2}\right)^2 + \frac{Y^2}{1-e^2} = \frac{h^2e^2}{(1-e^2)^2}.$$

Proof: Again, we use the earlier note which tells us a point $P = (X, Y)$ lies on a conic if and only if it satisfies $\operatorname{dist}(P, F) = e \operatorname{dist}(P, L)$. All we need do is apply the relevant distance formulae and tweak things with a bit of algebraic manipulation:

$$\begin{aligned}
\operatorname{dist}(P, F) = e \operatorname{dist}(P, L) &\Leftrightarrow \sqrt{X^2 + Y^2} = e|X + h| \\
&\Leftrightarrow X^2 + Y^2 = e^2(X^2 + 2Xh + h^2) \\
&\Leftrightarrow X^2(1 - e^2) + Y^2 - 2Xhe^2 = h^2e^2 \\
&\Leftrightarrow X^2 + \frac{Y^2}{1-e^2} - \frac{2Xhe^2}{1-e^2} = \frac{h^2e^2}{1-e^2} \\
&\Leftrightarrow \left(X - \frac{he^2}{1-e^2}\right) - \frac{h^2e^4}{(1-e^2)^2} + \frac{Y^2}{1-e^2} = \frac{h^2e^2}{1-e^2} \\
&\Leftrightarrow \left(X - \frac{he^2}{1-e^2}\right) + \frac{Y^2}{1-e^2} = \frac{h^2e^4}{(1-e^2)^2} + \frac{h^2e^2}{1-e^2} \\
&\Leftrightarrow \left(X - \frac{he^2}{1-e^2}\right) + \frac{Y^2}{1-e^2} = \frac{he^2}{(1-e^2)^2}.
\end{aligned}$$

□

Note: The only assumption in Proposition 4.9 is $e \neq 1$, so it will apply also when $e > 1$.

Corollary 4.10 *The* standard form of an ellipse *is* $x^2/a^2 + y^2/b^2 = 1$ *for all* $a > b > 0$.

- *Its centre is* $(0, 0)$.
- *Its eccentricity is* $e = \sqrt{1 - b^2/a^2}$.
- *Its focus is the point* $(ae, 0)$.
- *Its directrix is the line* $x = a/e$.

Proof: Let $a = he/(1-e^2)$ in Proposition 4.9. Then, the equation of the conic becomes $(X - ae)^2 + Y^2/(1-e^2) = a^2$. Thus, dividing through by a^2 and setting $x = X - ae$ and $y = Y$ and $b = a\sqrt{1-e^2}$, we get the intended form. From this final equation, we can immediately rearrange to get the formula for the eccentricity as stated above. Finally, per what we just said,

$$\begin{aligned}(X-ae)^2 + Y^2/(1-e^2) = a^2 &\Leftrightarrow x^2 + y^2/(1-e^2) = a^2\\ &\Leftrightarrow x^2(1-e^2) + y^2 = a^2(1-e^2)\\ &\Leftrightarrow x^2 - e^2x^2 + y^2 = a^2 - a^2e^2\\ &\Leftrightarrow x^2 - a^2e^2 + y^2 = e^2x^2 + a^2\\ &\Leftrightarrow x^2 - 2aex - a^2e^2 + y^2 = e^2x^2 - 2aex + a^2\\ &\Leftrightarrow (x-ae)^2 + y^2 = e^2(x - a/e)^2\\ &\Leftrightarrow \sqrt{(x-ae)^2 + y^2} = e\big|x - a/e\big|.\end{aligned}$$

Therefore, we see that the distance between (x, y) and the point $(ae, 0)$ is precisely e multiplied the distance between (x, y) and the line $x = a/e$. By definition of a conic, this means that the focus is $(ae, 0)$ and the directrix is $x = a/e$. □

Exercise 13 Carefully read through each line of algebra in the above proof and make sure you can justify what is being done at each stage to get to the next line. If you find it easy, it is good practice; if you find it difficult, it is even better practice.

Remark 4.11 Looking at Corollary 4.10, it is worth pointing out that we can give an alternate description of the focus and directrix as follows: in the fifth line of algebra in the proof of said corollary, we decided to subtract $2aex$ from both sides to give us a neat factorisation. However, we could equally have decided to **add** $2aex$, which again gives us a neat factorisation and ultimately yields

$$\sqrt{(x+ae)^2 + y^2} = e\big|x + a/e\big|.$$

Thus, the standard ellipse has another focus at $(-ae, 0)$ and another directrix described by $x = -a/e$. Hence, there are two foci and two directrices. This actually captures the fact that the ellipse is symmetric about the y-axis.

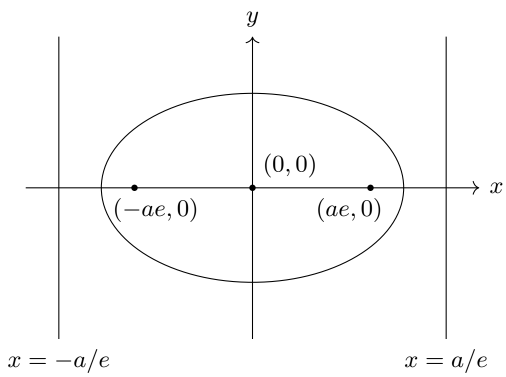

Figure 23: The standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with directrices.

Definition 4.12 Consider the standard ellipse $x^2/a^2 + y^2/b^2 = 1$ for $a > b > 0$.

- The major axis is the line segment from $(-a, 0)$ to $(a, 0)$.
- The minor axis is the line segment from $(0, -b)$ to $(0, b)$.
- The semi-major axis is half the major axis, thus having length a.
- The semi-minor axis is half the minor axis, thus having length b.

Proposition 4.13 (Parametric Equation of an Ellipse) *Let $a > b > 0$. A point $P = (x, y)$ lies on the ellipse $x^2/a^2 + y^2/b^2 = 1$ if and only if there exists $t \in [0, 2\pi)$ such that $P = (a\cos t, b\sin t)$.*

Proof: ($\Rightarrow$) Let $P(x, y)$ lie on the ellipse and consider the new coordinates $(X, Y) = (x, ay/b)$. Then, the equation for our ellipse in the (x, y)-plane becomes $X^2 + Y^2 = a^2$ in the (X, Y)-plane; we have transformed it to a circle with centre the origin and radius a. By Proposition 3.16, we know that $(X, Y) = (a\cos t, a\sin t)$ where $t \in [0, 2\pi)$. Reversing our coordinate transformation, we see that this is equivalent to $(x, y) = (a\cos t, b\sin t)$.

($\Leftarrow$) Given $(x, y) = (a\cos t, b\sin t)$ for some $t \in [0, 2\pi)$, it is clear that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2(t)}{a^2} + \frac{b^2\sin^2(t)}{b^2} = \cos^2(t) + \sin^2(t) = 1. \qquad \square$$

Remark 4.14 We say that a coordinate transformation is a rigid transformation if it preserves distances between points. These shall appear in a new context in Chapter **??**. Note that the transformation in the proof of the forward direction of Proposition 4.13 is **not** rigid because it changed the shape from an ellipse to a circle, thus some distances will be altered. We claim, and will later see, that the only rigid transformations in $\mathbb{R}^2$ are translations, rotations and reflections (or combinations of the three).

Note: A circle can be considered as a sort-of *degenerate* ellipse, in that if we allow $a = b$, we would get the equation of a circle. In this case, the eccentricity $e = 0$. As such, there is no description for a directrix, analogous to that in Corollary 4.10.

Example 4.15 Suppose we wish to sketch the ellipse described by $9x^2 - 72x + 16y^2 + 32y + 16 = 0$. The goal is to again get squared x-terms and squared y-terms on one side, from which it will be clear what the values of a and b are. To achieve this, one can complete the square in both x and y and then divide through to get fractions with squares on their denominators:

$$9\left((x-4)^2 - 16\right) + 16(y+1)^2 = 0 \quad \Leftrightarrow \quad \frac{(x-4)^2}{4^2} + \frac{(y+1)^2}{3^2} = 1.$$

Therefore, defining $(X, Y) = (x - 4, y + 1)$ will give us the standard equation of an ellipse in the (X, Y)-plane. We proceed as in Example 4.6, transforming the foci, directrices, major axis and minor axis to determine how to sketch them.

- Its eccentricity is $e = \sqrt{1 - 3^2/4^2} = \sqrt{7}/4$.
- Its centre is $(X, Y) = (0, 0)$, which is to say $(x, y) = (4, -1)$.
- Its foci are $(X, Y) = (\pm\sqrt{7}, 0)$, which is to say $(x, y) = (4 \pm \sqrt{7}, -1)$.
- Its directrices are $X = \pm 16/\sqrt{7}$, which is to say $x = 4 \pm 16/\sqrt{7}$.

Finally, note that the axes crossings are $(4 \pm 8\sqrt{2}/3, 0)$ and $(0, -1)$.

Exercise 14 Actually sketch the ellipse discussed in Example 4.15.

[**Hint:** We approximate these in the (x, y)-plane: the foci are $(6.65, -1)$ and $(1.35, -1)$; the directrices are $x = 10.05$ and $x = -2.05$; the x-axis crossings are $(7.77, 0)$ and $(0.23, 0)$.]

Conic III: The Hyperbola

Definition 4.16 A hyperbola is the conic obtained when the eccentricity $e > 1$.

Again, deriving the standard form of a hyperbola isn't as simple as for the parabola. However, Proposition 4.9 already does the heavy-lifting and we see that the standard form drops out almost immediately from this result. Before proceeding, we will 'define' the following: an *asymptote* of a curve in $\mathbb{R}^2$ is a line such that the distance between the line and the curve approaches zero as either the x-coordinate or the y-coordinate tends to infinity. Essentially, this is a line which a curve gets arbitrarily close to (but does not touch) at infinity.

Corollary 4.17 *The standard form of a hyperbola is $x^2/a^2 - y^2/b^2 = 1$ for all $a, b > 0$.*

- *Its centre is $(0,0)$.*
- *Its eccentricity is $e = \sqrt{1 + b^2/a^2}$.*
- *Its vertices are the points $(\pm a, 0)$.*
- *Its foci are the points $(\pm ae, 0)$.*
- *Its directrices are the lines $x = \pm a/e$.*
- *Its asymptotes are the lines $x/a \pm y/b = 0$.*

Proof: Let $a = -he/(1-e^2)$ in Proposition 4.9. Then, the equation of the conic becomes $(X + ae)^2 + Y^2/(1-e^2) = a^2$. Thus, dividing through by a^2 and setting $x = X + ae$ and $y = Y$ and $b = a\sqrt{e^2 - 1}$, we get the intended form. From this final equation, we can immediately rearrange to get the formula for the eccentricity as stated above. The rest of the proof follows identically to that of Corollary 4.10. □

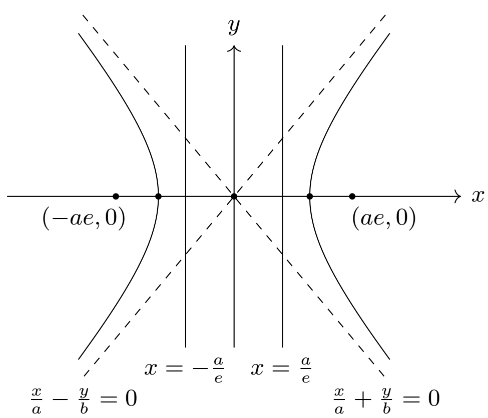

Figure 24: The standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with directrices and asymptotes.

Remark 4.18 Let's say something more on the asymptotic behaviour of the hyperbola. Indeed, we can see that the standard form of a hyperbola is the difference of two squares. As such, it can be factorised (this was something you may remember from high school):

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 1.$$

In this factorised form, we can see the asymptote equations appearing. We can now justify why the asymptotes are given by the formulae in Corollary 4.17 by considering two cases.

- If x, y are large with the **same** sign, $x/a - y/b$ is small; we are 'close to' $x/a - y/b = 0$.
- If x, y are large with **opposite** signs, $x/a + y/b$ is small; we are 'close to' $x/a + y/b = 0$.

Proposition 4.19 (Parametric Equations of a Hyperbola) *Let $a, b > 0$. A point $P = (x, y)$ lies on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ if and only if one of the following is true:*

(i) *There exists $t \in [-\pi, \pi] \setminus \{\pi/2\}$ such that $P = (a \sec t, b \tan t)$.*

(ii) *There exists $t \in \mathbb{R}$ such that $P = (a \cosh t, b \sinh t)$.*

Proof: Omitted; we leave this as an exercise for later (once we have introduced the hyperbolic functions cosh and sinh and the other trigonometric functions sec and cosec etc.). □

Example 4.20 Suppose we wish to sketch the hyperbola described by $9x^2 + 36x - 16y^2 + 160y = 220$. The goal is to again get squared x-terms and squared y-terms on one side, from which it will be clear what the values of a and b are, precisely as is done when sketching an ellipse. Indeed, we complete the square much the same as in Example 4.15:

$$9\left((x+2)^2 - 4\right) - 16\left((y+5)^2 - 25\right) = 220 \quad \Leftrightarrow \quad \frac{(y-5)^2}{3^2} - \frac{(x+2)^2}{4^2} = 1.$$

To ensure we get our standard form (which is x-squared terms minus y-squared terms), our new coordinates are $(X, Y) = (y - 5, x + 2)$. We proceed as in Example 4.15, transforming the foci, directrices, etc. to determine how to sketch them.

- Its eccentricity is $e = \sqrt{1 + 3^2/4^2} = 5/3$.
- Its centre is $(X, Y) = (0, 0)$, which is to say $(x, y) = (-2, 5)$.
- Its vertices are $(X, Y) = (\pm 3, 0)$, which is to say $(x, y) = (-2, 5 \pm 3)$.
- Its foci are $(X, Y) = (\pm 5, 0)$, which is to say $(x, y) = (-2, 5 \pm 5)$.
- Its directrices are $X = \pm 9/5$, which is to say $y = 5 \pm 9/5$.
- Its asymptotes are $X/3 \pm Y/4 = 0$, which is to say $3x + 4y - 14 = 0$ and $3x - 4y + 26 = 0$.

Finally, note that the axes crossings are $(0, 5 \pm \sqrt{45}/2)$ and $(10/3, 0)$ and $(-22/3, 0)$.

Exercise 15 Actually sketch the hyperbola discussed in Example 4.20.

[**Hint:** You can approximate most things using that $1/3 \approx 0.33$ and $2/3 \approx 0.67$; the y-axis crossings are approximately $(0, 8.35)$ and $(0, 1.65)$.]

Note: There is a special case known as a rectangular hyperbola, that is where the values $a = b$ in the standard form of the hyperbola. It is so named because this guarantees that the asymptotes are perpendicular (a fact we can see if we look at the equation of each asymptote). Moreover, the eccentricity of a rectangular hyperbola $e = \sqrt{2}$.

Conic Summary

We now provide a helpful table which collects the key information about each conic. Following this, we will look at how to classify conics in a relatively easy way. Note that in Examples 4.6, 4.15, 4.20, we were given a conic in a non-standard form but we were pre-told what conic it was. In the 'real-world', we may not be so lucky. Thus, we want theory to tell us what conic section we are working with. Before we get ahead of ourselves, here is the summary we promised.

	Parabola	**Ellipse**	**Hyperbola**
Eccentricity Value	$e = 1$	$0 < e < 1$	$e > 1$
Equation	$y^2 = 4ax$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Restrictions	$a \neq 0$	$a > b > 0$	$a, b > 0$
Centre	—	$(0, 0)$	$(0, 0)$
Vertices	$(0, 0)$	$(\pm a, 0)$	$(\pm a, 0)$
Foci	$(a, 0)$	$(\pm ae, 0)$	$(\pm ae, 0)$
Directrices	$x = -a$	$x = \pm \frac{a}{e}$	$x = \pm \frac{a}{e}$
Asymptotes	—	—	$\frac{x}{a} \pm \frac{y}{b} = 0$
Eccentricity Formula	—	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 + \frac{b^2}{a^2}}$

Table 1: A summary of the important properties of each conic.

Now, we move towards the classification of conics. To do this, we consider the so-called *degree* of a curve in two variables, namely x and y. Of particular interest are degree two curves.

Example 4.21 A curve of degree one in two variables is a curve of the form $ax + by + c = 0$, where $a, b, c \in \mathbb{R}$ such that a and b are not both zero. The degree can be thought of as the highest power of either x or y present in the equation of the curve which does not have a coefficient of zero (of course, $ax + by + c = 0$ is the same as $0y^5 + ax + by + c = 0$ but both have degree one).

Note: In Remark 3.3, we noted a degenerate situation wherein we reduce to a degree zero curve if $a = 0 = b$. This describes $\mathbb{R}^2$ when $c = 0$ and this describes $\emptyset$ when $c \neq 0$.

We now prove a critical result which provides us with the classification theorem.

Proposition 4.22 *Consider an arbitrary degree two curve in two variables of the form*

$$ax^2 + 2hxy + by^2 + fx + gy + c = 0.$$

If there is a rotation of the (x, y)-axes to the (X, Y)-axes such that the above becomes

$$AX^2 + 2HXY + BY^2 + FX + GY + C = 0,$$

then the so-called discriminant *remains unchanged, that is*

$$AB - H^2 = ab - h^2.$$

Proof: Suppose we rotate the (x, y)-axes by angle α. Then, Proposition 3.23 implies that

$$\begin{aligned} X &= x\cos(\alpha) + y\sin(\alpha), \\ Y &= -x\sin(\alpha) + y\cos(\alpha) \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} x &= X\cos(\alpha) - Y\sin(\alpha), \\ y &= X\sin(\alpha) + Y\cos(\alpha). \end{aligned}$$

Substituting these into the equation of the degree two curve in two variables gives us

$$\begin{gathered} a(X\cos\alpha - Y\sin\alpha)^2 + 2h(X\cos\alpha - Y\sin\alpha)(X\sin\alpha + Y\cos\alpha) + b(X\sin\alpha + Y\cos\alpha)^2 \\ + f(X\cos\alpha - Y\sin\alpha) + g(X\sin\alpha + Y\cos\alpha) + c = 0. \end{gathered}$$

Despite how messy this is, we can 'simplify' so that it has the form

$$\begin{gathered} (a\cos^2\alpha + 2h\cos\alpha\sin\alpha + b\sin^2\alpha)X^2 + 2\Big((b-a)\cos\alpha\sin\alpha + h(\cos^2\alpha - \sin^2\alpha)\Big)XY \\ + (a\sin^2\alpha - 2h\cos\alpha\sin\alpha + b\cos^2\alpha)Y^2 + \cdots = 0, \end{gathered}$$

where we don't even bother working out the extra terms (hence the $+\cdots$) because we already have everything we need to compute the discriminant (note that the discriminant only regards the stuff in front of x^2, y^2, xy and X^2, Y^2, XY). These are our new coefficients:

$$\begin{aligned} A &= a\cos^2\alpha + 2h\cos\alpha\sin\alpha + b\sin^2\alpha, \\ B &= a\sin^2\alpha - 2h\cos\alpha\sin\alpha + b\cos^2\alpha, \\ H &= (b-a)\cos\alpha\sin\alpha + h(\cos^2\alpha - \sin^2\alpha). \end{aligned}$$

Consequently, we can see that

$$\begin{aligned} AB - H^2 &= (ab - h^2)(\sin^4 \alpha + 2\sin^2 \alpha \cos^2 \alpha + \cos^4 \alpha) \\ &= (ab - h^2)(\sin^2 \alpha + \cos^2 \alpha)^2 \\ &= ab - h^2. \end{aligned}$$

□

Exercise 16 Read through the above proof slowly and try to convince yourself of the final calculation, namely the part where we show that $AB - H^2 = ab - h^2$.

Before pressing on to the classification, we need to note some of the degenerate conics, that is where $h = 0$ in the degree two curve in two variables. Indeed, we have the following:

(i) $x^2 + y^2 = 1$. (Circle)

(ii) $x^2 + y^2 = 0$. (Point)

(iii) $x^2 + y^2 = -1$. (Empty Set)

(iv) $x^2 = 1$. (Two Parallel Lines)

(v) $x^2 = 0$. (One Line)

(vi) $x^2 = -1$. (Empty Set)

(vii) $x^2 - y^2 = 0$. (Two Perpendicular Lines)

If it isn't clear, note that (i)–(iii) are ellipses, (iv)–(vi) are parabolas and (vii) is a hyperbola.

Proposition 4.23 *Consider the curve $ax^2 + by^2 + fx + gy + c = 0$ with a, b not both zero.*

(i) *If a, b have the same sign, then it is an ellipse.*

(ii) *If a, b have opposite signs, then it is a hyperbola.*

(iii) *If one of a, b is zero, then it is a parabola.*

Proof: We first complete the square and get an alternate formulation of the curve as follows:

$$a(x + p)^2 + b(y + q)^2 = r,$$

for $p, q, r \in \mathbb{R}$. If we set $X = x + p$ and $Y = y + q$, the curve can be written as $aX^2 + bY^2 = r$ in (X, Y)-coordinates , from which we can look at a number of cases.

(i) Let a, b have the same sign. If $r < 0$, this describes the empty set as in (iii) above. If $r = 0$, this describes a single point as in (ii). If $r > 0$ and $a = b$, this describes a circle as in (i) above. In all other instances, it describes an ellipse.

(ii) Let a, b have opposite signs. If $r = 0$, this describes two perpendicular lines as in (vii) above. In all other instances, it describes a hyperbola.

(iii) Let $a = 0$ and $b \neq 0$. Here, we can complete the square in y and divide through by b. Changing variables will give us $Y^2 = pX + q$ for some $p, q \in \mathbb{R}$. If $p = 0$, this describes the empty set (for $q < 0$) as in (vi) above, one line (for $q = 0$) as in (v) above or two parallel lines (for $q > 0$) as in (iv) above. In all other instances, it describes a parabola.

(iv) Let $a \neq 0$ and $b = 0$. This is near-identical to the previous case. □

Lemma 4.24 *Consider the curve $ax^2 + 2hxy + by^2 + fx + gy + c = 0$ with $a \neq b$ and let α be the angle defined as follows:*

$$\tan(2\alpha) = \frac{2h}{a - b}.$$

Rotating the (x, y)-axes about the origin by angle α means the curve can be re-written as

$$AX^2 + BY^2 + FX + GY + C = 0.$$

Proof: Proceeding similarly as the start of the proof of Proposition 4.22, we see that the second-order terms (i.e. those involving x^2, y^2, xy) are given by

$$AX^2 + 2\left((b - a)\cos\alpha\sin\alpha + h(\cos^2\alpha - \sin^2\alpha)\right)XY + BY^2,$$

where the precise formulae for $A, B \in \mathbb{R}$ isn't important and the (X, Y)-axes are obtained by rotating the (x, y)-axes about the origin by angle α. Now, the XY-term is eliminated when

$$\begin{aligned} 2(b - a)\cos\alpha\sin\alpha + 2h(\cos^2\alpha - \sin^2\alpha) = 0 &\Leftrightarrow (b - a)\sin(2\alpha) + 2h\cos(2\alpha) = 0 \\ &\Leftrightarrow (a - b)\sin(2\alpha) = 2h\cos(2\alpha) \\ &\Leftrightarrow \sin(2\alpha)/\cos(2\alpha) = 2h/(a - b) \\ &\Leftrightarrow \tan(2\alpha) = 2h/(a - b), \end{aligned}$$

by Proposition 2.42. Note that if $\cos(2\alpha) = 0$, then $\sin(2\alpha) \neq 0$ but because we assumed that $a \neq b$, there are no solutions for $\cos(2\alpha) = 0$, that is we can assume that $\cos(2\alpha) \neq 0$, allowing us to divide through above. □

Note: Should $a = b$, it is still possible to eliminate the mixed term by choosing $\alpha = \pi/4$.

Theorem 4.25 (Conic Classification Theorem) *All curves* $ax^2+2hxy+by^2+fx+gy+c=0$ *describe a (possibly degenerate) conic. Moreover, the discriminant classifies the conic:*

(i) *If* $ab-h^2>0$*, the conic is an ellipse.*

(ii) *If* $ab-h^2=0$*, the conic is a parabola.*

(iii) *If* $ab-h^2<0$*, the conic is a hyperbola.*

Proof: By Lemma 4.24, we can choose a rotation of the (x,y)-axes to eliminate the mixed term, transforming the equation of the curve to $AX^2+BY^2+FX+GY+C=0$. If $A=0=B$, there are no terms of degree two, so this cannot happen. As such, at least one of A,B is non-zero. By Proposition 4.22, the discriminant is preserved, that is $ab-h^2=AB$. The result then follows from Proposition 4.23. □

Example 4.26 We classify the conic described by $13x^2+6\sqrt{3}xy+7y^2-16=0$. Indeed, we can see that the discriminant of this conic is $13(7)-(3\sqrt{3})^2=64>0$, so this is an ellipse.

Exercise 17 Following from Example 4.26, apply the rotation in Lemma 4.24 to transform the ellipse to its standard form and use this to determine its eccentricity.

We finish the discussion on conics by stating, but not proving, a neat formula in terms of polar coordinates. Although we haven't used polar coordinates to describe too many curves, we will see more examples in Chapter **??**.

Proposition 4.27 *Consider a conic with eccentricity* e*, defined by a directrix* L *and focus* F *at a distance* h *away from* L*. Then, in terms of polar coordinates with the pole at* F *and the polar axis perpendicular to* L*, the equation of the conic is* $r(1-e\cos\theta)=eh$*.*

5 Three-Dimensional Geometry

We are now ready to make the jump up another dimension. This section concerns the discussion of lines and planes in three-dimensional space. We will also study vectors for the first time, which will allow us to get more abstract in Chapter **??** and more practical in Chapter **??**.

Note: For shorthand, we write $\mathbb{R}^3 := \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ to mean the three-dimensional space. This notation is consistent with the notation used for the two-dimensional plane.

Definition 5.1 The line through (x_1, y_1, z_1) in the direction (u, v, w) is the subset

$$L = \{(x, y, z) : (x, y, z) = (x_1, y_1, z_1) + t(u, v, w) \text{ for } t \in \mathbb{R}\}.$$

Remark 5.2 In Definition 5.1, the point (x_1, y_1, z_1) is a fixed point and we think of (u, v, w) as a *vector*, that is specifying a direction through which the line passes through the chosen fixed point. Finally, we have the *parameter* $t \in \mathbb{R}$, so this is really a parametrisation similar to what we saw for the conics. Moreover, we may make the notation more compact in the following way:

$$X = (x, y, z), X_1 = (x_1, y_1, z_1), U = (u, v, w), \quad \text{so the line is} \quad X = X_1 + tU.$$

Exercise 18 Write the equation $y = mx + c$ in a similar way to Definition 5.1.

[**Hint:** The line $y = mx + c$ describes the set of points $(x, y) = (t, mt + c)$ for $t \in \mathbb{R}$.]

Lemma 5.3 *Let $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ be distinct points in $\mathbb{R}^3$. Then, the line through X_1 and X_2 is given by $X = X_1 + t(X_2 - X_1)$.*

Proof: We know from Definition 5.1 that the line is given by $X = X_1 + sU$ for some parameter $s \in \mathbb{R}$ and direction U. Since X_2 lies on the line, there exists $p \in \mathbb{R}$ such that $X_2 = X_1 + pU$, which is to say that $U = \frac{1}{p}(X_2 - X_1)$. Consequently, if we reparametrise $t = s/p$, then the equation of the line becomes precisely what is needed: $X = X_1 + t(X_2 - X_1)$. □

Exercise 19 Find the equation of the line through $(1, 1, 1)$ and $(1, -1, -1)$. If possible, reparametrise so that the direction part of the equation is as 'simple' as possible.

Definition 5.4 A plane is a subset of the form

$$\Pi = \{(x, y, z) : ax + by + cz = d \text{ for } a, b, c, d \in \mathbb{R} \text{ with } a, b, c \text{ not all zero}\}.$$

Note: There is no set notation for a plane, since P is often used for points. Some people use π, but we tweak this notation and use Π to distinguish it from other objects later.

Example 5.5 Consider the points $(0, 1, 1), (2, 0, -1), (1, -1, 1)$. How can we determine the equation of the plane containing these points? Essentially, it amounts to substituting these points into the equation $ax + by + cz = d$ and solving to find the values of a, b, c, d. If we substitute these points in, we get the following equations:

$$\begin{aligned} b + c &= d, \\ 2a - c &= d, \\ a - b + c &= d. \end{aligned}$$

Solving these simultaneously (if you can't do this yet, it will be treated in detail in Chapter **??**), we see that $a = 4d/5, b = 2d/5, c = 3d/5$. We can now choose **any** value for d and we automatically get values for a, b, c. Of course, a sensible choice is $d = 5$. In this case, the equation we get is

$$4x + 2y + 3z = 5.$$

Exercise 20 Find the equation of the plane containing $(1, 2, 3), (0, 1, 1), (2, 2, 0)$.

Of course, as in $\mathbb{R}^2$, we have a natural notion of distance in $\mathbb{R}^3$. Namely, the distance between two points is the length of the line segment between them. However, we want to determine a neat formula which gives us that distance in terms of the coordinates of each point.

Theorem 5.6 *Let $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ be points in $\mathbb{R}^3$. Then,*

$$\mathrm{dist}(X_1, X_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Note: We denote $\mathrm{dist}(X_1, X_2) = |X_2 - X_1|$ and we call $\mathrm{dist}(O, X_1) = |X_1|$ the magnitude.

Proof: Consider the case $X_1 = (0, 0, 0)$. Then, we must show that $\mathrm{dist}(O, X_2) = \sqrt{x_2^2 + y_2^2 + z_2^2}$. We consider two additional points: $X_3 = (x_2, y_2, 0)$ and $X_4 = (0, y_2, 0)$ as in Figure 25.

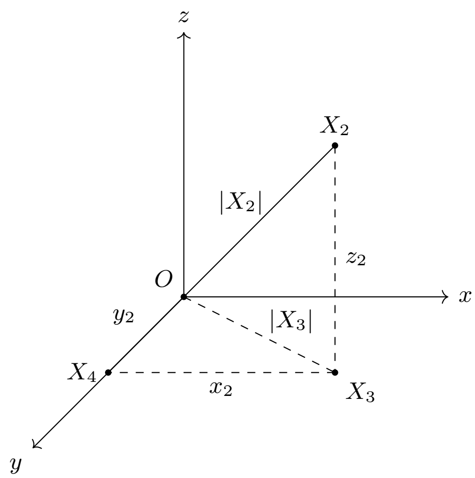

Figure 25: The construction to compute $\mathrm{dist}(O, X_2)$ in $\mathbb{R}^3$.

Since OX_3X_4 is a right-angled triangle, Pythagoras' Theorem implies that $|X_3|^2 = x_2^2 + y_2^2$. Since OX_2X_3 is a right-angled triangle, Pythagoras' Theorem implies that $|X_2| = |X_3|^2 + z_2^2$. Substituting in the previous formula tells us that $|X_2| = x_1^2 + y_1^2 + z_1^2$, as required. The general case follows by translating X_1 to the origin and back again. □

Definition 5.7 Let $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$. The dot product is

$$X_1 \cdot X_2 = x_1x_2 + y_1y_2 + z_1z_2.$$

Remark 5.8 The dot product is also known as the scalar product; this is because it produces a so-called *scalar*, that is a number in this context. Hence, this is an example of a function which takes two *vectors* and from them produces a number. To emphasise this fact, it may be called a *scalar-valued function*.

Example 5.9 The dot product $(1, 3, -4) \cdot (-2, 0, 2) = 1(-2) + 3(0) - 4(2) = -2 - 8 = -10$.

Exercise 21 Compute the dot product $(2, -5, 9) \cdot (-1, -2, 3)$.

Lemma 5.10 *Let $X_1, X_2, X_3 \in \mathbb{R}^3$. We have the following properties of the dot product.*

(i) $X_1 \cdot X_1 = |X_1|^2$.

(ii) $(-X_1) \cdot X_2 = -(X_1 \cdot X_2) = X_1 \cdot (-X_2)$.

(iii) $X_1 \cdot X_2 = X_2 \cdot X_1$.

(iv) $(X_1 + X_2) \cdot X_3 = (X_1 \cdot X_3) + (X_2 \cdot X_3)$.

Sketch of Proof: (i) This is trivial; recall that the magnitude $|X_1| = \sqrt{x_1^2 + y_1^2 + z_1^2}$ by Theorem 5.6, so squaring it gives us precisely $x_1^2 + y_1^2 + z_1^2$, which is Definition 5.7 with $X_2 = X_1$.

(ii) Just apply the definition with $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$.

(iii) This follows from the fact that multiplication of two real numbers is commutative, that is for all $a, b \in \mathbb{R}$, we have $ab = ba$.

(iv) Just apply the definition with $X_1 = (x_1, y_1, z_1)$, $X_2 = (x_2, y_2, z_2)$ and $X_3 = (x_3, y_3, z_3)$. □

Theorem 5.11 *Let $X_1, X_2 \in \mathbb{R}^3$ be distinct from the origin and $\theta \in [0, \pi]$ be the angle between the line segments OX_1 and OX_2. Then, the dot product satisfies*

$$X_1 \cdot X_2 = |X_1||X_2| \cos(\theta).$$

Proof: If you are unfamiliar with vectors, skip this proof and return to it after Chapter **??**. Let $\mathbf{a}$ be the vector OX_1 and $\mathbf{b}$ be the vector OX_2. Then, the vector X_1X_2 is defined as $\mathbf{c} = \mathbf{b} - \mathbf{a}$. If $\mathbf{a}$ and $\mathbf{b}$ are parallel, meaning $\mathbf{a} = k\mathbf{b}$ for some $k \in \mathbb{R}$, then the result is immediate since $\theta = 0$. Otherwise, we turn to the set-up pictured in Figure 26.

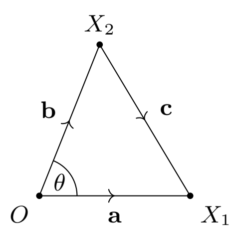

Figure 26: The vectors OX_1 and OX_2 in $\mathbb{R}^3$.

If we apply the Cosine Rule (Theorem 2.43) to triangle OX_1X_2, we see that

$$\begin{aligned}
|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta) &= |\mathbf{c}|^2 \\
&= |\mathbf{b} - \mathbf{a}|^2 \\
&= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\
&= \mathbf{b} \cdot (\mathbf{b} - \mathbf{a}) - \mathbf{a} \cdot (\mathbf{b} - \mathbf{a}) \\
&= (\mathbf{b} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{a}) \\
&= |\mathbf{a}|^2 - 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2,
\end{aligned}$$

where we use the properties in Lemma 5.10. Consequently, we can rearrange this to get

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta).$$

All that remains is to note that we can alternatively denote $\mathbf{a} = X_1$ and $\mathbf{b} = X_2$. □

Exercise 22 For each line of the Cosine Rule argument in the above proof, state which of the properties (i) to (iv) in Lemma 5.10 is used to get each equality.

[**Hint:** The first equality is the Cosine Rule, the second equality is the fact $\mathbf{c} = \mathbf{b} - \mathbf{a}$, and so forth.]

Note: The angle between vectors X_1 and X_2 is defined herein as the angle between OX_1 and OX_2, where we regard X_1 and X_2 as points at the end of the line segments.

Definition 5.12 Two vectors X_1 and X_2 are orthogonal if $X_1 \cdot X_2 = 0$.

Example 5.13 Suppose we wish to determine the vectors that are orthogonal to $(0, 2, -1)$; this amounts to solving $(0, 2, -1) \cdot (x, y, z) = 0$, which is to say that $2y - z = 0$. Hence, the set of vectors orthogonal to the given one is $(x, y, 2y)$ for all $x, y \in \mathbb{R}$.

Definition 5.14 The angle between two lines, namely $X = X_1 + tU_1$ and $X = X_2 + tU_2$, in $\mathbb{R}^3$ is defined as the angle between their directions U_1 and U_2.

Example 5.15 Consider the lines defined parametrically as follows:

$$(x, y, z) = (1 + 2t, 1 - 2t, 1 + 4t) \quad \text{and} \quad (x, y, z) = (2 - 2t, 3 + 3t, 7 + 7t).$$

The first job is to notice that their directions are $(2, -2, 4) = (1, -1, 2)$, because we can divide out any common multiples, and $(-2, 3, 7)$, respectively. Applying Theorem 5.11 tells us that

$$(1, -1, 2) \cdot (-2, 3, 7) = \big|(1, -1, 2)\big|\big|(-2, 3, 7)\big| \cos(\theta) \quad \Leftrightarrow \quad 9 = \sqrt{372} \cos(\theta),$$

which we can rearrange to see that $\theta = \cos^{-1}(9/\sqrt{372}) \approx 1.09$ (in radians, as per usual).

Definition 5.16 Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. The cross product is

$$A \times B = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

Remark 5.17 The dot product is also known as the vector product; this is because it produces a vector. Hence, this is an example of a function which takes two *vectors* and from them produces another vector. To emphasise this fact, it may be called a *vector-valued function*.

Note: We can 'build' $\mathbb{R}^3$ from the three vectors $(1,0,0),(0,1,0),(0,0,1)$ in that any point $(x,y,z) \in \mathbb{R}^3$ can be written as $x(1,0,0)+y(0,1,0)+z(0,0,1)$. This is an example of a set of so-called *basis vectors*, something we will see much more of later. This particular set is called the standard basis of $\mathbb{R}^3$, and we denote these vectors by $\mathbf{i},\mathbf{j},\mathbf{k}$. Using these, we can re-write the cross product in terms of a matrix, specifically a determinant:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

If you haven't seen matrices, don't worry; just use the (nasty) formula in Definition 5.16.

Exercise 23 Compute the cross product of the vectors $A=(0,1,1)$ and $B=(2,0,-1)$.

Lemma 5.18 *Let $A,B \in \mathbb{R}^3$. We have the following properties of the dot product.*

(i) $A \cdot (A \times B) = 0$.
(ii) $B \cdot (A \times B) = 0$.
(iii) $A \times B = -B \times A$.
(iv) $|A \times B|^2 = |A|^2|B|^2 - (A \cdot B)^2$.

Remark 5.19 Properties (i) and (ii) shows that the vector $A \times B$ is actually orthogonal to both of A and B; this is the purpose of the cross product. Moreover, property (iii) is an example of *anti-commutativity* and it implies that $A \times A = 0$. In particular, we have $A \times kA = 0$ for any $k \in \mathbb{R}$. Thus, the cross product of parallel vectors is zero.

Note: Orthogonal vectors are unique up to multiplication by a non-zero scalar. Indeed, if A and B are non-parallel vectors and N is a non-zero vector orthogonal to both of them, it follows that $N = k(A \times B)$ for some $k \in \mathbb{R}$.

The purpose of this discussion is to now simplify our description of a plane and providing an alternate (and easier) way to compute the equation of a plane.

Definition 5.20 Let Π be a plane. A normal vector to the plane is a non-zero vector N such that $N \cdot (P_1 - P_2) = 0$ for any two points $P_1, P_2 \in \Pi$.

Lemma 5.21 *The vector (a,b,c) is normal to the plane Π given by $ax+by+cz=d$.*

Proof: Let $N=(a,b,c)$, which means the equation of the plane can be written $N\cdot P=d$ where $P=(x,y,z)$ is a general point. If we have $P_1,P_2\in\Pi$, this means that $N\cdot P_1=d$ and $N\cdot P_2=d$. Hence, subtracting the second from the first and using Lemma 5.10 gives $N\cdot(P_1-P_2)=0$. □

Theorem 5.22 *Let $P_1,P_2,P_3\in\Pi$ be distinct non-collinear points in a plane. Then, any normal vector N to the plane is a non-zero multiple of $(P_1-P_2)\times(P_1\times P_3)$.*

Proof: Since the three points are distinct, we know that P_1-P_2 and P_1-P_3 are non-zero. Since the points are non-collinear, the angle between these vectors is not 0 or π. Thus, we have $N\cdot(P_1-P_2)=0=N\cdot(P_1-P_3)$ per the proof of Lemma 5.21. Then, the fact that any normal is a non-zero multiple of this is a consequence of the previous note. □

As a corollary, we see that (a,b,c) is a non-zero multiple of $(P_1-P_2)\times(P_1\times P_3)$. Because multiplying the equation of a plane by a non-zero number does not change the plane it describes, the equation of the plane can be written in the form

$$\alpha x+\beta y+\gamma z=\delta,\qquad \text{where } (P_1-P_2)\times(P_1\times P_3)=(\alpha,\beta,\gamma).$$

Example 5.23 Consider the three points $(1,-3,4),(0,5,-2),(1,7,2)\in\Pi$ in some plane Π and a line L parametrised as $(x,y,z)=(3,1,7)+t(1,-8,5)$ for $t\in\mathbb{R}$. We can find the equation of the plane by the method described above. Moreover, we can deduce the angle between the plane and the line.

(a) First, we find a normal vector N to the plane by computing the following cross product:

$$\big((1,-3,4)-(0,5,-2)\big)\times\big((1,-3,4)-(1,7,2)\big)=(1,-8,6)\times(0,-10,2)=(44,-2,-10).$$

Therefore, the equation of the plane is $44x-2y-10z=\delta$, where we can find the relevant δ by substituting in a point we know is on the plane, e.g. $(1,-3,4)$. Indeed, we see that $\delta=10$. We can divide through by two to get a slightly simpler final equation: $22x-y-5z=5$.

(b) As for the angle between Π and L, this is obtained by finding the angle between the normal to the plane and the line and subtracting this from $\pi/2$ (as explained in the next note). The angle $\alpha\in(0,\pi/2)$ between N and L is

$$\alpha=\cos^{-1}\left(\frac{(1,-8,5)\cdot(22,-1,-5)}{\big|(1,-8,5)\big|\big|(22,-1,-5)\big|}\right)=\cos^{-1}\left(\frac{5}{\sqrt{90}\sqrt{510}}\right)\approx 1.547.$$

Hence, the angle between the plane and line is $\theta = \pi/2 - \alpha \approx 0.02$.

Note: If we think of a plane Π with normal N and a line L through the plane, we can find the angle $\alpha \in (0, \pi/2)$ between the line and normal, but since the normal forms a right angle with the plane, subtracting this from $\pi/2$ will give the angle $\theta \in (0, \pi/2)$ between the line and plane. This is pictured in Figure 27 below.

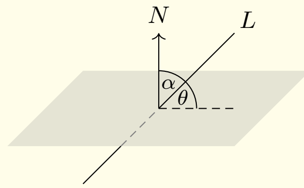

Figure 27: Finding the angle between a line L and a plane Π.

Remark 5.24 One can define a plane parametrically. Indeed, if X_1 is a point on the plane, then the equation can be written $X = X_1 + tU + sV$ where U and V are non-parallel vectors and $s, t \in \mathbb{R}$ are the parameters. The normal to this plane is the vector $U \times V$.

Exercise 24 Consider $A = (1, 1, 1), B = (1, -1, -1), C = (-1, 1, -1), D = (-1, -1, 1)$.

(i) State the equation of the line L through A and B.
[**Note:** This is your solution to Exercise 19.]

(ii) Find the equation of the plane Π containing B, C, D.

(iii) Determine the angle between Π and L.

6 Polyhedra

We end the discussion on elementary geometry with an introduction to three-dimensional shapes called polyhedra. First, we need to discuss what a polygon is (the two-dimensional analogue of a polyhedron). In fact, Exercise 24 contained a hidden polyhedron called a regular tetrahedron in that this is the shape with vertices $ABCD$.

Definition 6.1 A polygon is one closed path in the plane consisting of line segments (edges) intersecting only at endpoints (vertices) with no successive line segments parallel.

Example 6.2 We can see polygons in Figure 28(a) and non-polygons in Figure 28(b).

(a) Examples of polygons. (b) Non-examples of polygons.

Figure 28: Two examples and two non-examples of polygons.

Exercise 25 Explain why each of the shapes in Figure 28(b) is not a polygon.

We have already met a number of polygons in our mathematical careers:

- A polygon with three sides is called a triangle.
- A polygon with four sides is called a quadrilateral.
- A polygon with five sides is called a pentagon.
- A polygon with six sides is called a hexagon.

Note: A polygon with n sides is henceforth simply referred to as an n-gon.

Definition 6.3 A polygon is called convex if the line segment between any two points on the polygon's edge lies in the interior of the polygon.

Example 6.4 Looking at Figure 28(a), the first polygon (a triangle) is convex, whereas the second polygon (a hendecagon, or 11-gon) is non-convex. Note that there exists a convex 11-gon; we are not saying that **all** 11-gons are non-convex, only the one pictured in the figure. Note that we only show one line segment for the triangle but the point is it is true for **any** pair of points on the edge.

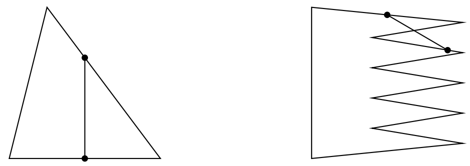

Figure 29: Convexity and non-convexity in Figure 28(a).

Remark 6.5 The idea of convexity can be extended to sets in general, as is discussed much later in Chapter **??**. The concept is the same, in that a set is called convex if a 'line segment' can be drawn between elements of the set and the entire segment lies in the interior of the set. This is actually an abstract idea but the picture to have in mind is that of the polygon situation in Definition 6.3.

Theorem 6.6 *The interior angles of a convex n-gon sum to $(n-2)\pi$.*

Proof: Choose a point X in the interior of the n-gon and draw line segments from X to each of the n vertices of the polygon; this divides the shape into n triangles. Now, the sum of the angles in the n triangles is therefore equal to the sum of the interior angles plus the sum of the angles around X. We know from Theorem 2.26 that the sum of the angles in these triangles is $n\pi$. We also know that the sum of the angles around a point is 2π. Thus, if S is the sum of the interior angles in the n-gon, we have

$$n\pi = S + 2\pi \quad \Leftrightarrow \quad S = (n-2)\pi. \qquad \square$$

Note: We can drop the 'convex' hypothesis in Theorem 6.6; it is true for any n-gon.

Definition 6.7 A polygon is called regular if all side lengths are equal and all interior angles are equal.

Example 6.8 Both parts of Definition 6.7 are actually necessary.

(i) A polygon with equal side lengths but unequal interior angles: rhombus.

(ii) A polygon with equal interior angles but unequal side lengths: rectangle

Neither of these is a regular polygon (specifically a regular quadrilateral). The only such regular quadrilateral is the intersection of both, that is the square.

Corollary 6.9 (of Theorem 6.6) *One interior angle of a regular n-gon is $(n-2)\pi/n$.*

Proof: By Theorem 6.6, the interior angles sum to $(n-2)\pi$. Now, since the n-gon is regular, each interior angle is equal and there are n of them. As such, the result is obvious. $\square$

Exercise 26 The exterior angle of a polygon is the angle between any edge and the line extended from the next edge, as demonstrated for a regular hexagon in Figure 30 below.

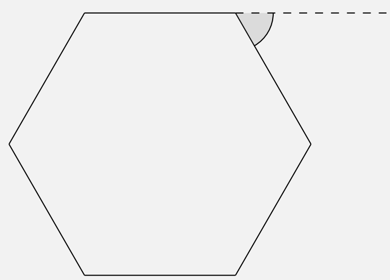

Figure 30: One of the six exterior angles of a hexagon.

Determine the value of one exterior angle of a regular n-gon.

We are now ready to re-enter the three-dimensional world. We will be first bombarded with a long definition covering the analogue of those made for polygons.

Definition 6.10 A polyhedron is a closed figure whose boundary consists of a finite number of non-parallel polygonal faces which intersect only at edges or vertices.

- It is convex if the line segment between any two points on the polyhedron's boundary lies in the interior of the polyhedron.
- It is regular if all faces are congruent polygons and the same number of faces meet at each vertex.

Definition 6.11 A polyhedron is a Platonic solid if it is both convex and regular.

Theorem 6.12 *There are exactly five Platonic solids, described in* Table 2 *with these:*

- v *is the total number of vertices.*
- e *is the total number of edges.*
- f *is the total number of faces.*
- p *is the number of edges on each face.*
- q *is the number of faces meeting each vertex.*

	v	e	f	p	q
Tetrahedron	4	6	4	3	3
Cube	8	12	6	4	3
Octahedron	6	12	8	3	4
Dodecahedron	20	30	12	5	3
Icosahedron	12	30	20	3	5

Table 2: A summary of the important properties of the Platonic solids.

Proof: Consider an arbitrary Platonic solid and suppose the faces are p-gons, that is there are p edges on each face, where q of them meet at a given vertex. Since the Platonic solids are convex (by definition), the angles at that vertex must sum to less than 2π. Well, there are q angles around each vertex and each of them is $(p-2)\pi/p$ by Corollary 6.9. Therefore,

$$\begin{aligned} & \left(\frac{p-2}{p}\pi\right) q < 2\pi \\ \Leftrightarrow \quad & (p-2)q < 2p \\ \Leftrightarrow \quad & pq - 2q - 2p < 0 \\ \Leftrightarrow \quad & pq - 2q - 2p + 4 < 4 \\ \Leftrightarrow \quad & (p-2)(q-2) < 4. \end{aligned}$$

Therefore, we need only go through all the cases where this inequality is satisfied by p and q. Because $p, q \geq 3$ are positive integers, the only option is to have $(p-2)(q-2) \in \{1, 2, 3\}$.

(i) Suppose $(p-2)(q-2) = 1$. This occurs if and only if $p = 3$ and $q = 3$. Building a convex polyhedron with three triangles meeting at a vertex will always result in a tetrahedron.

(ii) Suppose $(p-2)(q-2) = 2$. If $p = 4$ and $q = 3$, we will build a cube whereas if $p = 3$ and $q = 4$, we can only construct an octahedron.

(iii) Suppose $(p-2)(q-2)=3$. If $p=5$ and $q=3$, we will build a dodecahedron whereas if $p=3$ and $q=5$, this will result in an icosahedron. □

Theorem 6.13 (Euler's Formula) *Consider a polyhedron which can be 'deformed' into a sphere, such that it has v vertices, e edges and f faces. Then, $v-e+f=2$.*

Proof: Deferred; see Chapter **??** for a more general result. □

Note: Euler was one of the most influential mathematicians to exist. Along with a few other greats, you will see his name come up time and time again. As such, we may as well set the record straight on the pronunciation of his name. The uninitiated will claim it to be *yu-ler* (so it rhymes with 'ruler') but in fact it is **oi-ler** (so it rhymes with 'boiler').

Example 6.14 We will now demonstrate how one can use Euler's Formula to deduce that a given shape can **not** be 'deformed' (whatever that means) into a sphere. Indeed, consider the toroid in Figure 31 below.

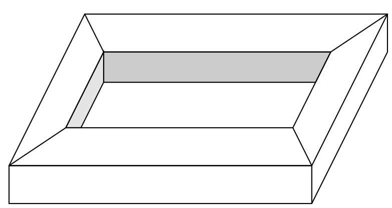

Figure 31: The toroid.

Here, $(v,e,f)=(16,32,16)$ and so $v-e+f=0\neq 2$; we cannot deform it into a sphere.

Remark 6.15 Really, Example 6.14 is our first look at a topological invariant. To set up Chapter **??** already, we mention now that there is a fundamental problem with deforming things that have 'holes' (whatever they are). Just on a basic, intuitive, common-sense level, we can see that the toroid in Figure 31 has a hole in the middle. When you think of a sphere, we know there is no hole, so this suggests that these two spaces are indeed different. This is what Euler's Formula captures, and this is something that can be extended to arbitrary dimensions (i.e. we talk about the Euler characteristic later, which is a generalisation of Euler's Formula and can be thought of as detecting 'n-dimensional holes', whatever they are).

Exercise 27 A geodesic sphere is a polyhedron constructed from triangles, with varying numbers meeting at each vertex and all vertices lying on the surface of a sphere. Consider a geodesic sphere which contains 100 triangles.

(i) How many edges does it have?

(ii) How many vertices does it have?

And here ends our journey into elementary geometry. As I'm sure can be inferred from the references in this chapter, the story continues. If you are interested in deforming shapes and finding invariants, then Chapter **??** is the place to go. For a more rigorous understanding of the geometry of curves and surfaces, we turn to Chapter **??** (but really we need Chapter **??** first, in order to hit the ground running when looking at the more advanced study of geometry).

7 Exercise Solutions

We provide detailed solutions to the exercises interwoven within each section of the module. Hopefully you have given these questions a try whilst on your learning journey with the module. But mathematics is difficult, so don't feel disheartened if you had to look up an answer before you knew where to begin (we have all done it)!

Solutions to Exercises in Section 2

Exercise 1 We discuss three points in Definition 2.6, but what can we say about the (simpler) situation of only two points: are two points always collinear? Are they always non-collinear? Give a one-sentence proof of the correct statement by using Axiom 2.3.

Solution: Two points are always collinear by the E1 axiom. □

Exercise 2 Prove that triangle congruence is an equivalence relation.

Solution: A triangle $ABC \simeq ABC$ via the bijection where $A \mapsto A$, $B \mapsto B$, $C \mapsto C$, meaning that triangle congruence is reflexive. Next, if $ABC \simeq DEF$, then there is a bijection f from the vertices A, B, C to the vertices D, E, F where corresponding edges/angles are equal. Since it is a bijection, it has an inverse f^{-1} (which is also a bijection) from the vertices D, E, F to the vertices A, B, C where corresponding edges/angles are equal. Hence, $DEF \simeq ABC$, meaning that triangle congruence is symmetric. Finally, if $ABC \simeq DEF$ and $DEF \simeq GHI$, there are bijections f from A, B, C to D, E, F and g from D, E, F to G, H, I. But now, Corollary **??** implies that $g \circ f$ from A, B, C to G, H, I is a bijection. Since each of these maps preserves corresponding edges/angles, the composition does so. Hence, $ABC \simeq GHI$, meaning that triangle congruence is transitive. Thus, it is an equivalence relation. □

Exercise 3 We have seen that the conditions SSS, SAS and ASA guarantee congruence. Can we conclude the same about 'AAA', that is does having all three angles in common imply that two triangles are congruent? If so, prove it. If not, give a counterexample.

Solution: There are triangles which share the same angles but are not congruent, meaning that AAA isn't a valid congruence condition (at least in Euclidean geometry; we can discuss the meaning of this later). Indeed, we can see an example of this in Figure 32 below. □

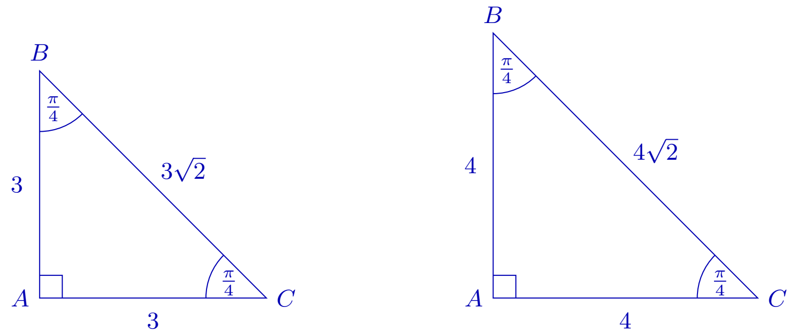

Figure 32: Non-congruent triangles with identical angles.

> **Exercise 4** State and prove the converse to Proposition 2.25.
>
> [**Hint:** Consider the angles $\alpha_1, \delta_1, \delta_2$ and combine Proposition 2.21 with Corollary 2.24.]

Solution: The converse is this: "let L_1 and L_2 be distinct lines and K a transversal. If the corresponding angles are equal, then L_1 and L_2 are parallel". We will now prove this. By assumption, we know that $\delta_1 = \delta_2$. But because angles on a straight line sum to π, we also have that $\alpha_1 + \delta_1 = \pi$. As such, we know that $\alpha_1 + \delta_2 = \pi$. By Proposition 2.21, vertically-opposite angles are equal, so $\gamma_1 = \alpha_1$. Again using that angles on a line sum to π, we know that $\alpha_2 + \delta_2 = \pi$, so this means that $\gamma_1 = \alpha_1 = \alpha_2$. Hence, we have a pair of alternate angles (α_2, γ_1) being equal. Corollary 2.24 therefore implies that L_1 and L_2 are parallel. □

> **Exercise 5** State and prove the triangle version of Proposition 2.34.
>
> [**Hint:** Use Proposition 2.34; draw a picture and tweak it to make it look like Figure 11.]

Solution: Let ABC and ABD be two triangles sharing the base AB, as in Figure 33.

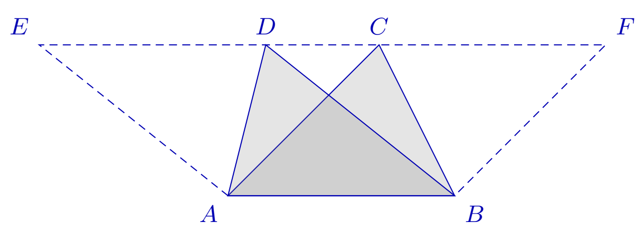

Figure 33: The triangles ABC and ABD.

We can construct a picture resembling Figure 11 by drawling the line (segment) AE that is parallel to BD and BF that is parallel to AC. In this way, we get two parallelograms $ABDE$

and $ABFC$ that share the same base and have the same height. By Proposition 2.34, they have the same area. Because AD is a diagonal of the parallelogram $ABDE$, it bisects it (Corollary 2.33). The same holds for the diagonal BC in parallelogram $ABFC$; the bisected regions are precisely the triangles ABC and ABD. □

Exercise 6 A trapezium is a four-sided shape with one pair of parallel edges, one such example is the shape $DEFG$ in Figure 14, with parallel sides DE and FG. The area of a trapezium is computed as follows:

(i) Add the lengths of the two parallel sides.
(ii) Multiply this by the (perpendicular) height.
(iii) Divide the result by two.

Now you are equipped with this, give an alternate proof of Pythagoras' Theorem (the forward direction) by computing the area of Figure 14, as we did for Figure 13 above.

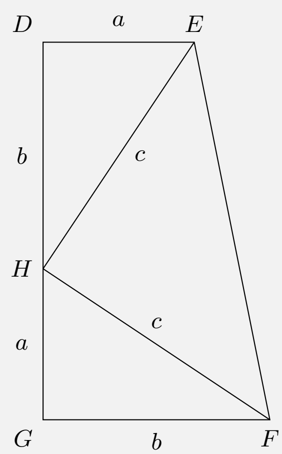

Figure 7.14: The trapezium for an alternate proof of Pythagoras' Theorem.

Solution: Following what the question says, the area of $DEFG$ is computed directly as

$$\frac{1}{2}(a+b)(a+b).$$

On the other hand, we can compute the areas of the constituent shapes, that is of triangles DEH, FGH and EFG. We then equate to the above area and rearrange. Indeed, this method yields the following expression for the area:

$$\frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2.$$

As such, equating and multiplying everything by two gives us

$$(a+b)(a+b) = 2ab + c^2 \quad \Leftrightarrow \quad a^2 + 2ab + b^2 = 2ab + c^2 \quad \Leftrightarrow \quad a^2 + b^2 = c^2. \qquad \square$$

Exercise 7 Determine what happens in the degenerate case $\alpha = \pi/2$.

[**Hint:** You can work out the value of $\cos(\pi/2)$ from the identity $\cos(\pi - \theta) = -\cos(\theta)$.]

Solution: Per the hint, we know that $\cos(\pi - \pi/2) = -\cos(\pi/2)$, where we take $\theta = \pi/2$. This is equivalent to $\cos(\pi/2) = -\cos(\pi/2)$, which can mean only that $\cos(\pi/2) = 0$. Therefore, when $\alpha = \pi/2$, the Cosine Rule reduces to Pythagoras' Theorem. $\square$

Exercise 8 Demonstrate that the equation $ax + by + c = 0$ can describe both our usual line $y = mx + p$ and our vertical line $x = k$. In other words, choose values for a, b, c so that the equation rearranges to get $y = mx + p$ (and again so that it rearranges to $x = k$).

Solution: Suppose we have a line of the form $y = mx + p$. Then, we can take $(a, b, c) = (m, -1, p)$ in the general equation of the line. This will give us $mx - y + p = 0$, which is clearly a rearrangement of the initial equation of the line.

As for a vertical line $x = k$, it is quite clear that $(a, b, c) = (1, 0, -k)$ will do the trick; this gives us $x + 0y - k = 0$ which is, again, an obvious rearrangement of the initial line. $\square$

Exercise 9 Prove the distance between two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is given by

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

[**Hint:** This is an application of Pythagoras' Theorem; draw a picture.]

Solution: We first draw a picture of two arbitrary points in $\mathbb{R}^2$, as in Figure 3.

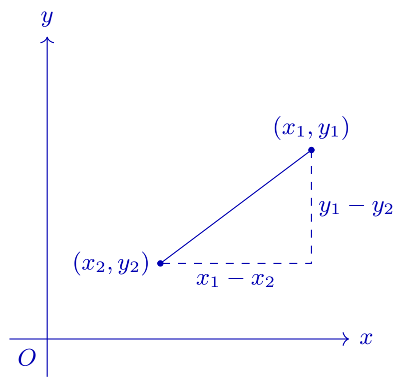

Figure 3: The distance between (x_1, y_1) and (x_2, y_2).

The distance between the points is the (solid) line segment in the above picture, which we can complete to a right-angled triangle. Then, Pythagoras' Theorem implies that the length of the hypotenuse, that is the distance we want, is precisely $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. □

Exercise 10 Prove (or justify some of) Proposition 3.12.

[**Hint:** (iii) and (iv) follow from the *old* definitions; (v) and (vi) follow from Figure 17; (vii) is Pythagoras' Theorem; (viii) is simply a compound-angle formula.]

Solution: Firstly, (i) and (ii) are clear from the circle interpretation; if we traverse an angle and then add 2π, that is just doing one full rotation on top of it, so we reach the same point. This is called the periodicity of sine and cosine.

As for (iii) and (iv), the hint tells us that these are a result of the old definitions. Another way to see it is more geometric and uses Figure 17. Indeed, take the point $(\cos\theta, \sin\theta)$. We first invert θ to get $-\theta$ (i.e. we reflect across the x-axis) and then we add π to get $\pi - \theta$ (i.e. move around by one half rotation). If we do that, we end up at a point with the same y-coordinate as we started, meaning $\sin(\pi - \theta) = \sin(\theta)$. However, the new point will have the same x-coordinate except with a minus sign, which means $\cos(\pi - \theta) = \cos(\theta)$.

We have already justified (v) and (vi) in the geometric interpretation of (iii) and (iv).

If we complete the point in Figure 17 to a right-angled triangle, we see that it has side lengths 1 (hypotenuse), $\cos\theta$ (horizontal) and $\sin\theta$ (vertical). Thus, Pythagoras' Theorem implies the result immediately: $\sin^2(\theta) + \cos^2(\theta) = 1$.

Finally, (viii) is a result of the compound-angle formula $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ from the note just before Theorem 3.6; we need only substitute $x = \theta$, $y = \pi/2$ and use that $\cos(\pi/2) = 0$ (proved in Exercise 7) and $\sin(\pi/2) = 1$. □

Exercise 11 Express the polar point $(2, \pi)$ in Cartesian coordinates. Furthermore, express the Cartesian point $(0, 0)$ in polar coordinates. Is your answer unique?

Solution: The polar point $(2, \pi)$ will have Cartesian coordinates $x = 2\cos(\pi)$ and $y = 2\sin(\pi)$, that is $(-2, 0)$. As for converting the Cartesian point $(0, 0)$ into polar coordinates, note that Lemma 3.21 does not provide us with the answer. However, we can think of the origin and the pole as being the same point, so we know that it will have polar coordinates $(0, \theta)$ for any θ. Consequently, it isn't (even remotely) unique. □

Note: Because (r, θ) measures how far we 'stick out' from the pole and how far around we have rotated, we write the origin in polar coordinates as $(0, \theta)$ because we aren't 'sticking out' at all, and thus it doesn't matter how much we rotate because we are already restricted to being at the pole by $r = 0$.

Exercise 12 Actually sketch the parabola discussed in Example 4.6.

[**Hint:** Although we didn't do so in the above example, you should also label on where the parabola intercepts the y-axis, that is substitute in $x = 0$ and solve.]

Solution: We first determine the axes intercepts. Indeed, the x-intercept occurs when $y = 0$. Substituting this into the original equation of the parabola gives us

$$0^2 - 6(0) + 3x = 10 \quad \Leftrightarrow \quad x = \frac{10}{3}.$$

As for the y-intercept, this occurs when $x = 0$. Substituting this gives us

$$y^2 - 6y + 3(0) = 10 \quad \Leftrightarrow \quad y^2 - 6y - 10 = 0 \quad \Leftrightarrow \quad y = 3 \pm \sqrt{19}.$$

We have everything needed to draw a sufficient sketch of the parabola, done in Figure 4. □

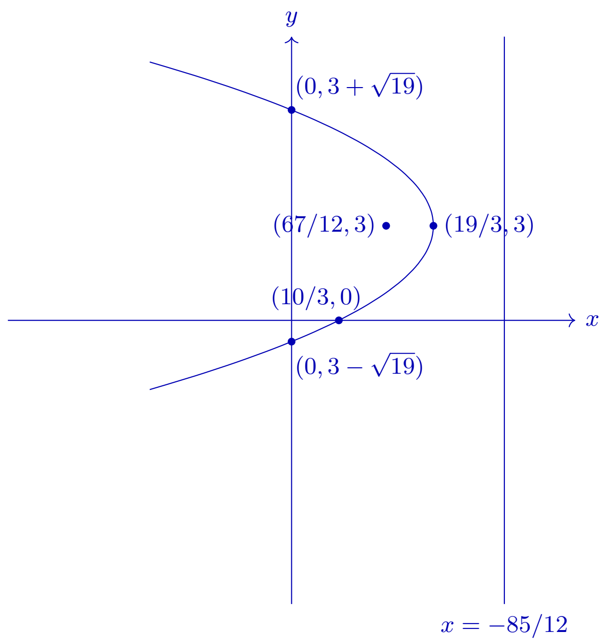

Figure 4: The parabola given by $y^2 - 6y + 3x = 10$.

Exercise 13 Carefully read through each line of algebra in the above proof and make sure you can justify what is being done at each stage to get to the next line. If you find it easy, it is good practice; if you find it difficult, it is even better practice.

Solution: Here is how we pass between the implications in the proof of Corollary 4.10:

(i) Line 1 to Line 2: Multiply by $1 - e^2$.

(ii) Line 2 to Line 3: Expand the brackets.

(iii) Line 3 to Line 4: Add e^2x^2 and subtract a^2e^2 on both sides.

(iv) Line 4 to Line 5: Subtract $2aex$ from both sides.

(v) Line 5 to Line 6: Factorise both sides in the x-variable.

(vi) Line 6 to Line 7: Take the square root of both sides. □

Note: Sometimes in a proof, something non-obvious will occur such as (iv) above: it seems almost random to subtract $2aex$ from both sides, but this is needed to make the factorisation in (v) work. This is the only reason such a random operation is carried out.

Exercise 14 Actually sketch the ellipse discussed in Example 4.15.

[**Hint:** We approximate these in the (x, y)-plane: the foci are $(6.65, -1)$ and $(1.35, -1)$; the directrices are $x = 10.05$ and $x = -2.05$; the x-axis crossings are $(7.77, 0)$ and $(0.23, 0)$.]

Solution: Again, we need to compute the intercepts of the ellipse with each axis. As in the solution to Exercise 12, we substitute $x = 0$ to determine the y-axis intercepts:

$$16y^2 + 32y + 16 = 0 \quad \Leftrightarrow \quad y^2 + 2y + 1 = 0 \quad \Leftrightarrow \quad y = -1.$$

Similarly, we substitute $y = 0$ to determine the x-axis intercepts:

$$9x^2 - 72x + 16 = 0 \quad \Leftrightarrow \quad x = 4 \pm \frac{8\sqrt{2}}{3}.$$

We have everything needed to draw a sufficient sketch of the ellipse, done in Figure 5. □

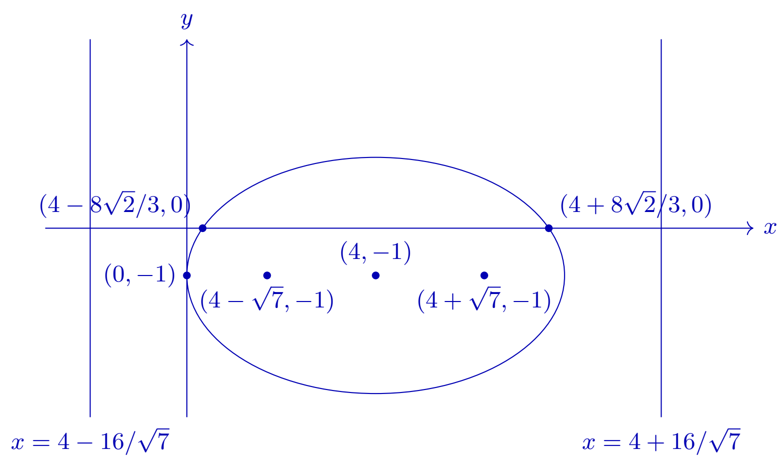

Figure 5: The ellipse given by $9x^2 - 72x + 16y^2 + 32y + 16 = 0$.

Exercise 15 Actually sketch the hyperbola discussed in Example 4.20.

[**Hint:** You can approximate most things using that $1/3 \approx 0.33$ and $2/3 \approx 0.67$; the y-axis crossings are approximately $(0, 8.35)$ and $(0, 1.65)$.]

Solution: Firstly, because we chose coordinates where X is in terms of y and Y is in terms of X, this amounts to a sort-of swapping of x and y in the picture of the standard hyperbola, that is Figure 24. Thus, where in the usual case our curves are vertical, this example will be of a hyperbola with horizontal curves. This is clear in Figure 6 below.

As usual, we will now determine the y-intercepts by substituting in $x = 0$:

$$-16y^2 + 160y = 220 \quad \Leftrightarrow \quad 4y^2 - 40y + 55 = 0 \quad \Leftrightarrow \quad y = 5 \pm \frac{3\sqrt{5}}{2}.$$

As for the x-intercepts, we substitute in $y = 0$:

$$9x^2 + 36x = 220 \quad \Leftrightarrow \quad (3x - 10)(3x + 22) = 0 \quad \Leftrightarrow \quad x = \frac{10}{3} \text{ and } x = -\frac{22}{3}.$$

We have everything needed to draw a sufficient sketch of the hyperbola, done in Figure 6. □

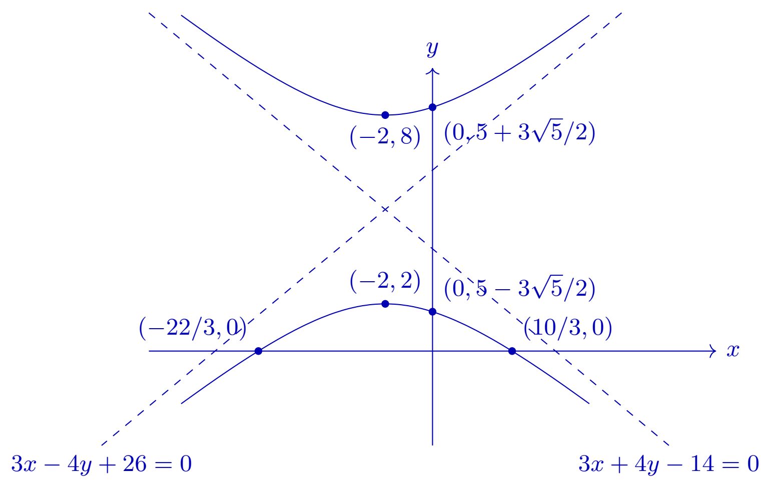

Figure 6: The hyperbola given by $9x^2 + 36x - 16y^2 + 160y = 220$.

Exercise 16 Read through the above proof slowly and try to convince yourself of the final calculation, namely the part where we show that $AB - H^2 = ab - h^2$.

Solution: Simply substitute these formulae for A, B, H, given in the proof of Proposition 4.22:

$$A = a\cos^2\alpha + 2h\cos\alpha\sin\alpha + b\sin^2\alpha,$$
$$B = a\sin^2\alpha - 2h\cos\alpha\sin\alpha + b\cos^2\alpha,$$
$$H = (b - a)\cos\alpha\sin\alpha + h(\cos^2\alpha - \sin^2\alpha).$$

Although it is messy algebra, we can see that

$$AB = (a\cos^2\alpha + 2h\cos\alpha\sin\alpha + b\sin^2\alpha)(a\sin^2\alpha - 2h\cos\alpha\sin\alpha + b\cos^2\alpha)$$

$$
\begin{aligned}
&= a^2\cos^2\alpha\sin^2\alpha - 2ah\cos^3\alpha\sin\alpha + ab\cos^4\alpha \\
&\quad + 2ah\cos\alpha\sin^3\alpha - 4h^2\cos^2\alpha\sin^2\alpha + 2bh\cos^3\alpha\sin\alpha \\
&\quad + ab\sin^4\alpha - 2bh\cos\alpha\sin^3\alpha + b^2\cos^2\alpha\sin^2\alpha
\end{aligned}
$$

and that

$$
\begin{aligned}
H^2 &= \Big((b-a)\cos\alpha\sin\alpha + h(\cos^2\alpha - \sin^2\alpha)\Big)^2 \\
&= b^2\cos^2\alpha\sin^2\alpha - 2ab\cos^2\alpha\sin^2\alpha + a^2\cos^2\alpha\sin^2\alpha \\
&\quad + 2bh\cos^3\alpha\sin\alpha - 2ah\cos^3\alpha\sin\alpha - 2bh\cos\alpha\sin^3\alpha + 2ah\cos\alpha\sin^3\alpha \\
&\quad + h^2\cos^4\alpha - 2h^2\cos^2\alpha\sin^2\alpha + h^2\sin^4\alpha.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
AB - H^2 &= ab\cos^4\alpha - h^2\cos^4\alpha + ab\sin^4\alpha - h^2\sin^4\alpha - 2h^2\cos^2\alpha\sin^2\alpha + 2ab\cos^2\alpha\sin^2\alpha \\
&= (ab - h^2)(\sin^4\alpha + 2\sin^2\alpha\cos^2\alpha + \cos^4\alpha),
\end{aligned}
$$

as was written in the proof of Proposition 4.22. Then, factorising and using $\sin^2\alpha + \cos^2\alpha = 1$ completes the calculation (they were the final two lines in the aforementioned proof). □

> **Exercise 17** Following from Example 4.26, apply the rotation in Lemma 4.24 to transform the ellipse to its standard form and use this to determine its eccentricity.

Solution: Recall that Example 4.26 gives us the conic described by $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$. Now, Lemma 4.24 states that a rotation by the angle α defined by $\tan(2\alpha) = 2h/(a-b)$, where a, b, h are coefficients of the conic equation, will transform the curve by removing the term with mixed variables. Indeed then, our curve is such that

$$
a = 13, \quad b = 7, \quad h = 3\sqrt{3}.
$$

Thus, we rotate by the angle α satisfying $\tan(2\alpha) = \sqrt{3}$, that is $\alpha = \pi/6$. Looking at the second line of the proof of Proposition 4.22, a direct consequence of Proposition 3.23 is that $x = X\cos(\alpha) - Y\sin(\alpha)$ and $y = X\sin(\alpha) + Y\cos(\alpha)$, where (X, Y)-coordinates are obtained by rotating the (x, y)-coordinates by α. In particular, our situation is this:

$$
x = \frac{\sqrt{3}}{2}X - \frac{1}{2}Y, \qquad y = \frac{1}{2}X + \frac{\sqrt{3}}{2}Y.
$$

Consequently, our equation becomes the following:

$$13\left(\frac{\sqrt{3}}{2}X-\frac{1}{2}Y\right)^2+6\sqrt{3}\left(\frac{\sqrt{3}}{2}X-\frac{1}{2}Y\right)\left(\frac{1}{2}X+\frac{\sqrt{3}}{2}Y\right)+7\left(\frac{1}{2}X+\frac{\sqrt{3}}{2}Y\right)^2-16=0.$$

If we expand out the brackets and simplify, we will get this form for the conic:

$$\frac{X^2}{1^2}+\frac{Y^2}{2^2}=1,$$

which we recognise as an ellipse (although it isn't in standard form because $a=1$ and $b=2$ but $a \not> b$). Nevertheless, if we want to be totally proper, we can make a simple second change of coordinates $(\mathcal{X},\mathcal{Y})=(Y,X)$ which essentially just 'swaps' the two letters around:

$$\frac{\mathcal{X}^2}{2^2}+\frac{\mathcal{Y}^2}{1^2}=1.$$

Hence, the eccentricity is

$$e=\sqrt{1-\frac{1^2}{2^2}}=\frac{\sqrt{3}}{2}.$$

□

Exercise 18 Write the equation $y=mx+c$ in a similar way to Definition 5.1.

[**Hint:** The line $y=mx+c$ describes the set of points $(x,y)=(t,mt+c)$ for $t\in\mathbb{R}$.]

Solution: Per the hint, we need only separate $(t,mt+c)$ into a sum of two pairs where one of them involves t and the other does not. Indeed, we can do this and it gives us

$$L=\{(x,y):(x,y)=t(1,m)+(0,c)\text{ for }t\in\mathbb{R}\}.$$

□

Exercise 19 Find the equation of the line through $(1,1,1)$ and $(1,-1,-1)$. If possible, reparametrise so that the direction part of the equation is as 'simple' as possible.

Solution: By Lemma 5.3, the line through these points is $(x,y,z)=(1,1,1)+t(0,-2,-2)$. Thus, we can reparametrise by choosing $s=-2t$ to get the 'simpler' equation as follows:

$$(x,y,z)=(1,1,1)+s(0,1,1),\quad s\in\mathbb{R}.$$

□

Exercise 20 Find the equation of the plane containing $(1,2,3),(0,1,1),(2,2,0)$.

Solution: We proceed as in Example 5.5. Indeed, we have to solve these simultaneously:

$$\begin{aligned} a + 2b + 3c &= d, \\ b + c &= d, \\ 2a + 2b &= d. \end{aligned}$$

We see that $b = d - c$ by the second equation. Hence, substituting this into the third equation gives $2a - 2c = -d$, which means that $2a = 2c - d$. If we double the first equation and substitute this in, we then know that

$$2a + 4b + 6c = 2d \quad \Leftrightarrow \quad (2c - d) + 4(d - c) + 6c = 2d \quad \Leftrightarrow \quad 2c - d + 4d - 4c + 6c = 2d.$$

This gives us $c = -d/4$. Consequently, we know that $b = 5d/4$ by the second equation and $a = -3d/4$ by the third equation. Thus, a sensible choice is $d = -4$ which means the equation of the plane will be $3x - 5y + z = -4$. □

Exercise 21 Compute the dot product $(2, -5, 9) \cdot (-1, -2, 3)$.

Solution: Well, $(2, -5, 9) \cdot (-1, -2, 3) = 2(-1) - 5(-2) + 9(3) = -2 + 10 + 27 = 35$. □

Exercise 22 For each line of the Cosine Rule argument in the above proof, state which of the properties (i) to (iv) in Lemma 5.10 is used to get each equality.

[**Hint:** The first equality is the Cosine Rule, the second equality is the fact $\mathbf{c} = \mathbf{b} - \mathbf{a}$, and so forth.]

Solution: Following the hint, these are how we get the equalities in the proof of Theorem 5.11:

- The third equality is property (i) from Lemma 5.10.
- The fourth equality is property (iv) from Lemma 5.10.
- The fifth equality is properties (ii) and (iv) from Lemma 5.10.
- The sixth equality is properties (i) and (ii) from Lemma 5.10. □

Exercise 23 Compute the cross product of the vectors $A = (0, 1, 1)$ and $B = (2, 0, -1)$.

Solution: Per Definition 5.16, the cross product is

$$A \times B = \big(1(-1) - 1(0), 1(2) - 0(-1), 0(0) - 1(2)\big) = (-1, 2, 2).$$ □

Note: Per the note after Definition 5.16 in the notes, we can compute the cross product as the following determinant (where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard basis vectors of $\mathbb{R}^3$):

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \mathbf{i}\left(1(-1) - 1(0)\right) - \mathbf{j}\left(0(-1) - 1(2)\right) + \mathbf{k}\left(0(0) - 1(2)\right),$$

which is just $(-1, 0, 0) + (0, 2, 0) + (0, 0, 2) = (-1, 2, 2)$, agreeing with the above method. As previously said, if matrices are new/you don't like them yet, ignore this note for now!

Exercise 24 Consider $A = (1, 1, 1), B = (1, -1, -1), C = (-1, 1, -1), D = (-1, -1, 1)$.

(i) State the equation of the line L through A and B.
[**Note:** This is your solution to Exercise 19.]

(ii) Find the equation of the plane Π containing B, C, D.

(iii) Determine the angle between Π and L.

Solution: (i) The equation of the line L is this: $(x, y, z) = (1, 1, 1) + s(0, 1, 1)$, for $s \in \mathbb{R}$.

(ii) First, we will find a normal vector N to the plane Π by computing the cross product of the vectors $B - C = (2, -2, 0)$ and $B - D = (2, 0, -2)$. Indeed, we can use the definition to see that

$$(B - C) \times (B - D) = (4, 4, 4).$$

Therefore, the vector $N = (1, 1, 1)$ will suffice as our normal to the plane. As such, the equation of the plane is $x + y + z = d$, where we can find d by substituting in one of our points B, C, D. Indeed, we easily see that Π is defined by $x + y + z = 3$.

(iii) Similar to Example 5.23, the angle between Π and L is given by subtracting $\pi/2$ from the following angle, which is the angle $\alpha \in (0, \pi/2)$ between N and L:

$$\alpha = \cos^{-1}\left(\frac{(0,1,1)\cdot(1,1,1)}{|(0,1,1)||(1,1,1)|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{3}}\right) \approx 0.6155.$$

Therefore, the angle between the plane and line is $\theta = \pi/2 - \alpha \approx 0.9553$. □

Exercise 25 Explain why each of the shapes in Figure 28(b) is not a polygon.

Solution: The left shape in Figure 28(b) is two closed paths, not one. The right shape in Figure 28(b) contains an intersection **not** at an endpoint. □

Exercise 26 The exterior angle of a polygon is the angle between any edge and the line extended from the next edge, as demonstrated for a regular hexagon in Figure 30 below.

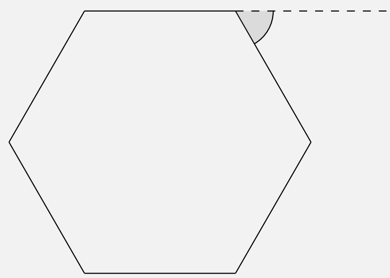

Figure 7.30: One of the six exterior angles of a hexagon.

Determine the value of one exterior angle of a regular n-gon.

Solution: The sum of one interior and one exterior angle is π, since angles on a line sum to π. Therefore, given that a regular n-gon has interior angles each of size $(n-2)\pi/n$ by Corollary 6.9, it follows that the value of one exterior angle of said polygon is

$$\pi - \frac{n-2}{n}\pi = \frac{2\pi}{n}.$$

□

Exercise 27 A geodesic sphere is a polyhedron constructed from triangles, with varying numbers meeting at each vertex and all vertices lying on the surface of a sphere. Consider a geodesic sphere which contains 100 triangles.

(i) How many edges does it have?

(ii) How many vertices does it have?

Solution: The number of edges is half the total number of sides, meaning there are 150 edges. Furthermore, we know there are 100 faces, by definition, so Euler's Formula implies that there are $2 + E - F = 52$ vertices. □