MATH1400 Modelling with Differential Equations

Cheatsheet

2022/23

This document collects together the important definitions and results presented throughout the lecture notes. The numbering used throughout will be consistent with that in the lecture notes.

Contents

1 Introduction

1.1 Differential Equations: The Basics

1.1.1 The Derivative

Definition The derivative of a function $f(x)$ at a particular value of x is the slope of the tangent to the curve at point $(x, f(x))$. Assuming the following limit exists, it is given by

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

Note: In the notes, they often call the function $y(x)$ instead of $f(x)$. We will start to do this also when we introduce differential equations, but the below results are probably easier to read if the functions are written $f(x)$ and $g(x)$, as opposed to $y(x)$ and $z(x)$.

Reminder: The following derivatives are well-known and may come in handy:

(i) For $f(x) = x^n$ with $n \in \mathbb{Z}$, we have $f'(x) = nx^{n-1}$.

(ii) For $f(x) = \sin(x)$, we have $f'(x) = \cos(x)$.

(iii) For $f(x) = \cos(x)$, we have $f'(x) = -\sin(x)$.

(iv) For $f(x) = \tan(x)$, we have $f'(x) = \sec^2(x)$.

(v) For $f(x) = e^x$, we have $f'(x) = e^x$.

(vi) For $f(x) = \log(x)$, we have $f'(x) = 1/x$, where log is the natural logarithm.

Proof: Use the limit definition above, possibly in conjunction with the theorem below. □

Theorem *Suppose we can differentiate the functions $f(x)$ and $g(x)$. Then, these are true:*

(i) $(f + g)'(x) = f'(x) + g'(x)$. (**Sum Rule**)

(ii) $(fg)(x) = f'(x)g(x) + f(x)g'(x)$. (**Product Rule**)

(iii) $(f/g)'(x) = \big(f'(x)g(x) - f(x)g'(x)\big)/g(x)^2$ **if** $g(x) \neq 0$. (**Quotient Rule**)

(iv) $(f \circ g)'(x) = f'(g(x))g'(x)$. (**Chain Rule**)

1.1.2 Dependent and Independent Variables

Definition An independent variable defines the 'space' on which things vary. A dependent variable are the variables that we solve for in a model, and vary as a function of the independent variables.

Note: We can have multiple dependent variables, an example being when we model the evolution of a two-dimensional position of a particle by $(x(t), y(t))$, depending on time t.

1.1.3 Ordinary Differential Equations

Definition An ordinary differential equation (ODE) is an equation that involves derivatives with respect to one independent variable. In other words, it contains derivatives only of the forms $y' := \mathrm{d}x/\mathrm{d}y$ and $y'' := \mathrm{d}^2x/\mathrm{d}y^2$ and $y''' := \mathrm{d}^3x/\mathrm{d}y^3$ and so forth.

Note: An ODE has an explicit solution if we can determine $y = f(x)$ satisfying it. Moreover, it may have an implicit solution, which is a relationship between x and y that **can't** be solved for y, i.e. an equation of the form $f(x, y) = C$ for some constant C.

1.1.4 Classification

The classification of an ODE can go a long way towards us eventually obtaining a solution. Indeed, the first step is to understand what it is we are dealing with. These are a good first few things to look for in our ODE to classify it:

- The order of the ODE.
- Whether it is linear or non-linear.
- Whether it is autonomous or non-autonomous.

Definition The order of an ODE is the largest number of times that the dependant variable is differentiated in the ODE.

For example, $y' = y$ is a first-order ODE, whereas $y''' - 3y' + 2 = x$ is a third-order ODE.

Definition An ODE is linear if it does **not** contain products or powers (or other non-linear functions) of the **dependent** variable and its derivatives. Otherwise, it is called non-linear.

For example, $y'' + 2y = -y'$ is linear, whereas $y''y' + \sqrt{y} = 7$ is non-linear.

Note: In a linear ODE, it is only the dependent variable that has big restrictions; the independent variable x can appear anywhere. For instance, this is the most general linear second-order ODE, where a, b, c, f are **any** given functions of the independent variable:

$$a(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = f(x).$$

Definition An ODE is autonomous if there is **no** explicit mention of the independent variable anywhere. Otherwise, it is called non-autonomous.

For example, $y'''' = y''$ is autonomous, whereas $x^2y' = \sin(x)$ is non-autonomous.

1.1.5 Verifying Solutions and Determining the ODE from a Given General Solution

Method – Verifying a Solution: To verify that a given function is a solution of an ODE, simply substitute the function in. This will involve differentiating the given function some number of times, but we can deal with this by using the facts stated in §1.1.1 above.

Definition A general solution to an ODE will involve arbitrary constants. The particular solution (given some extra data) will assign some definite values to each of the constants.

Note: The general solution to an n^{th}-order ODE involves n arbitrary constants $C_1, ..., C_n$.

Method – Isolate and Eliminate: Suppose we have a general solution $y(x)$ of an ODE and we wish to 'back-track' to obtain the ODE. There are two steps to follow:

(i) Isolate one of the constants by rearranging to make it the subject.
(ii) Eliminate it by differentiating the equation.
(iii) Repeat Steps (i) and (ii) until there are no constants left; what remains is the ODE.

1.2 Boundary Conditions

We use boundary conditions to get from a general solution of an ODE to a particular solution.

Definition For an ODE, a boundary condition is a specification of the value of the solution at a certain value of x. In the case that we treat the independent variable x as time, we call a boundary condition an initial condition when we fix x at the initialisation of our model (usually they have the form $y(0) = k$).

1.3 Philosophy of Mathematical Modelling

Mathematical modelling is a way to translate our understanding of something physical into the language of mathematics. Typically, it involves writing down a (set of) equation(s) to describe the behaviour of the physical system.

Definition An empirical model is often data-driven and uses statistical methods to infer relations between different variables in a system. On the other hand, a predictive model is one using prior knowledge and understanding of the mechanism that cause change in the system.

Note: Clearly then, if we are given some value of a quantity at a fixed time and we wish to predict its value in the future under some rate of change, this lends itself nicely to being modelled using a differential equation with boundary conditions.

2 Solutions of First-Order ODEs

2.1 Introduction

The general problem of solving a first-order ordinary differential equation takes the form

$$y' = f(y, x), \qquad \text{with } y(x_0) = y_0.$$

2.2 Separable First-Order ODEs

Definition A first-order ODE is separable if it can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)b(y),$$

where $a(x)$ and $b(y)$ are one-variable functions of x and y, respectively.

Method – Solving Separable ODEs: Suppose we have the separable ODE $y' = a(x)b(y)$.

(i) Divide both sides by $b(y)$.

(ii) Integrate both sides with respect to x (and include the constant of integration).

(iii) Next, change variables in the integral for y, meaning we have

$$\int \frac{1}{b(y)}\ \mathrm{d}y = \int a(x)\ \mathrm{d}x + C.$$

(iv) Perform the indefinite integrals and rearrange the answer for $y(x)$.

Note: It may **not** always be possible to evaluate the integrals written in Step (iii) above.

2.2.1 ODEs of Homogeneous Degree

Definition An ODE of homogeneous degree is a differential equation that can be given as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g\left(\frac{y}{x}\right).$$

Method – Solving ODEs of Homogeneous Degree: Suppose we have the following ODE of homogeneous degree: $y' = g(y/x)$. Define the new variable $v(x) := y(x)/x$.

(i) Substitute $y(x) = v(x)x$ into the ODE; this replaces ys and y's with vs and v's.

(ii) We obtain the following form of the ODE of homogeneous degree, which is separable:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{g(v) - v}{x}.$$

(iii) Solve this using the previous method and swap variables back to $y(x)$ at the end.

2.3 Exact ODEs

Definition A first-order ODE is called exact if it can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[H(x,y)\Big] = 0,$$

where $H(x,y)$ is a two-variable function dependent on both x and $y = y(x)$.

Method – Solving Exact ODEs: It is easy to find a solution of an exact ODE, because we can just integrate both sides with respect to x to obtain $H(x,y) = C$, where C is the constant of integration.

We would like to know if, given some first-order ODE, we can tell if it is possible to find a two-variable function $H(x,y)$ such that we can write the ODE in exact form. Even better, we would like an algorithm for finding said $H(x,y)$.

Lemma *Let $M(x,y) + N(x,y)y' = 0$ be a general first-order ODE, where M and N are given two-variable functions of x and y. Then, the ODE is exact if and only if*

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Proof: To show that the ODE is exact, we need to construct a two-variable function $H(x,y)$. Well, if we use a Multivariate Chain Rule on the left-hand side, we obtain the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[H(x,y)\Big] = \frac{\partial H}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}y',$$

the second equality using the fact that $\partial x/\partial x = 1$ and that $\partial y/\partial x = \mathrm{d}y/\mathrm{d}x = y'$. We know from the assumption that the ODE is equal to zero, so combining it with the above formula tells us

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}y' = 0.$$

Comparing this with the expression for the ODE in the statement of the lemma, we see that

$$\frac{\partial H}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial H}{\partial y} = N(x,y).$$

If we (partially) differentiate the first with respect to y and the second with respect to x, we get

$$\frac{\partial^2 H}{\partial y \partial x} = \frac{\partial M}{\partial y} \qquad \text{and} \qquad \frac{\partial^2 H}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

But the Mixed Derivatives Theorem tells us that $\partial^2 H/\partial y\partial x = \partial^2 H/\partial x \partial y$, so both the guys above are equal; this is precisely what we set out to prove. We can trace this argument in reverse to conclude that the ODE is exact if we have $\partial M/\partial y = \partial N/\partial x$. □

Reminder: There are many versions of the Chain Rule depending on how many variables we have. The one we used in the above proof was this, where f is a function of k variables which we call $v_1, ..., v_k$ but **these** themselves are each a function of the single variable x:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[f\left(v_1(x), ..., v_k(x)\right)\Big] = \frac{\partial f}{\partial v_1}\frac{\partial v_1}{\partial x} + \cdots + \frac{\partial f}{\partial v_k}\frac{\partial v_k}{\partial x}.$$

Method – Writing an ODE in Exact Form: Consider the ODE $M(x, y) + N(x, y)y' = 0$.

(i) Solve the equation $\partial H/\partial x = M$ for H. To do this, we integrate with respect to x, treating all ys as constant. Because of this, the constant of integration is itself any function of y, so we have $H = A(x, y) + f(y)$.

(ii) Solve the equation $\partial H/\partial y = N$ for H. To do this, we integrate with respect to y, treating all xs as constant. Because of this, the constant of integration is itself any function of x, so we have $H = B(x, y) + g(x)$.

(iii) Determine $f(y)$ and $g(x)$ so these solutions are the same; this gives us our H.

2.3.1 Integrating Factors

Definition If a non-exact ODE can be made exact simply by multiplying it by a function R (possibly of both x and y), we call said function an integrating factor.

Remark If the integrating factor $R = R(x)$ or $R = R(y)$, then there are systematic approaches that can determine R explicitly. The case where $R = R(x, y)$ is a two-variable function is **not** considered in the MATH1400 course.

Method – Finding an Integrating Factor: Suppose we know the integrating factor $R(x)$ exists and we have an autonomous ODE (no explicit mention of x).

(i) Multiply the ODE by the integrating factor $R(x)$.

(ii) Impose the exactness condition, namely $\partial M/\partial y = \partial N/\partial x$.

(iii) Simplify the equation in Step (ii) to get an ODE that R satisfies.

(iv) Solve the ODE in Step (iii) to obtain R.

2.4 Linear First-Order ODEs

Definition A linear first-order ODE is a differential equation of the form

$$y' + P(x)y = Q(x), \qquad \text{with } y(x_0) = y_0.$$

Proposition *A linear first-order ODE of the form* $y' + P(x)y = Q(x)$ *can be solved using the following integrating factor*

$$R(x) = \exp\left(\int P(x)\ \mathrm{d}x\right) = e^{\int P(x)\ \mathrm{d}x}.$$

Proof: Let us first multiply the ODE by $R(x)$ to obtain this equation:

$$R(x)y' + P(x)R(x)y = Q(x)R(x).$$

If we can pick R such that the left-hand side is the derivative of $R(x)y(x)$, the above becomes

$$R(x)y' + P(x)R(x)y = \frac{\mathrm{d}}{\mathrm{d}x}\Big[R(x)y(x)\Big] = R(x)y' + R'(x)y.$$

Comparing the far-left to the far-right, one concludes that we must have

$$\frac{\mathrm{d}R}{\mathrm{d}x} = R(x)P(x).$$

This is a separable ODE ($P(x)$ is given to us) which we solve for R using the standard method:

(i) Dividing both sides by R yields $\dfrac{1}{R}\dfrac{\mathrm{d}R}{\mathrm{d}x} = P(x)$.

(ii) Integrating with respect to x tells us that $\displaystyle\int \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}x}\ \mathrm{d}x = \int P(x)\ \mathrm{d}x$.

(iii) We change variables on the left-hand side to obtain $\displaystyle\int \frac{1}{R}\ \mathrm{d}R = \int P(x)\ \mathrm{d}x$.

(iv) We can perform this integral, giving us $\log\big(|R|\big) = \displaystyle\int P(x)\ \mathrm{d}x + K$.

Finally then, if we take the exponential of both sides, this provides us with a general solution

$$R(x) = \exp\left(\int P(x)\ \mathrm{d}x + K\right) = C\exp\left(\int P(x)\ \mathrm{d}x\right),$$

where $C \coloneqq \exp(K)$. Now, it really doesn't matter what C is (along as it is non-zero) since multiplication by any such R will make the ODE exact; we can set $C = 1$ (that is $K = 0$). □

Method – Solving Linear First-Order ODEs: Let us be given a linear first-order ODE.

(i) Write it in the form $y' + P(x)y = Q(x)$.
(ii) Compute the integrating factor $R(x) = \exp\big(\int P(x)\ \mathrm{d}x\big)$.
(iii) Multiply the whole ODE by $R(x)$ and use $R' = RP$ to write it as $\frac{\mathrm{d}}{\mathrm{d}x}[Ry] = RQ$.
(iv) Integrate both sides with respect to x (and include the constant of integration).
(v) Perform the indefinite integrals and rearrange the answer for $y(x)$.

2.4.1 Bernoulli's Equation

Definition For $n \neq 0, 1$, Bernoulli's Equation is this **non-linear** ODE: $y' + P(x)y = Q(x)y^n$.

Note: If $n = 0$, then the equation is linear and is identical to that which we considered just above. If $n = 1$, we can simply write it as $y' + \big(P(x) - Q(x)\big)\, y = 0$ which is again a linear first-order ODE. Therefore, the integrating factor method in either case will provide a solution.

Lemma *Bernoulli's Equation can be linearised by changing variables to $z(x) = y^{1-n}$.*

Proof: We need to show that making the change of variables above gives us a linear first-order ODE. Well, the first thing to note is that the (standard) Chain Rule implies

$$z' = (1-n)y^{-n}y'.$$

We can rearrange this for y' (and the change of variables formula for y) and substitute into Bernoulli's equation and simplify. A more economical way to do things is to divide both sides of Bernoulli's equation by y^n to obtain

$$y^{-n}y' + P(x)y^{1-n} = Q(x).$$

We can now almost immediately substitute z and z' into the above:

$$\frac{1}{1-n}z' + P(x)z = Q(x),$$

which is equivalent to $z' + (1-n)P(x)z = (1-n)Q(x)$, a linear first-order ODE for z. $\square$

Method – Solving Bernoulli's Equation: Consider $y' + P(x)y = Q(x)y^n$ for $n \neq 0, 1$.

(i) Linearise Bernoulli's Equation as in the proof above.
(ii) Solve the linear first-order ODE for z using the previous method.
(iii) Change variables back to the original y variable.

Note: In Bernoulli's Equation, we can allow $n \in \mathbb{R} \setminus \{0, 1\}$ to be (almost) any real number!

2.5 Visualising Solutions to First-Order ODEs: The Slope Field

Definition Suppose we have a first-order ODE of the form $y' = f(x, y)$. A slope field is a map which shows the rate of change of the solution for all values of x and y. It produces a 'flow' along which the solutions of the ODE travel.

Method – Plotting a Slope Field: Consider an ODE of the form $y' = f(x, y)$.

(i) Choose a point (x, y) in the xy-plane and substitute it into f.

(ii) From that point, draw an arrow whose slope is the value $f(x, y)$.

(iii) Repeat Steps (i) and (ii) with many (all) points in the xy-plane.

By definition of solutions of the ODE $y' = f(x, y)$, we can visualise its solutions as tracing lines in the slope field that lie tangent to the arrows along its path. Best of all, we can sketch the solution without solving the first-order ODE a priori!

Note: When plotting a slope field, first consider the function $f(y)$ which tells us how the slope will depend on y only. We call a solution an attractor if other flow lines (solutions) approach it at the limit; we call a solution a repeller if other flow lines (solutions) diverge away from it at the limit.

3 Developing and Constraining Models

3.1 Radioactive Decay

Atoms of a given radioactive isotope have a fixed percentage chance of decaying in any given time period. Given that a sufficiently-large number of atoms are considered as a mass, the number of decays that occur in a fixed time period is proportional to the number of atoms and the length of time. This motivates the discussion below.

3.1.1 Decay and Half-Life

Definition Let $N(t)$ be the number of atoms at time t and consider this number a short time later $N(t+\delta t)$. We call the positive number $k > 0$ the decay constant if it satisfies

$$N(t+\delta t) = N(t) - kN(t)\delta t.$$

Note: At time t, the number of atoms decreases by an amount proportional to (i) the number of current atoms $N(t)$ and (ii) the length of time δt over which the decay happens. This is captured by the equation in the above definition.

Proposition *The number of atoms $N(t)$ at time t is given by the equation*

$$N(t) = N_0 e^{-k(t-t_0)},$$

where we assume the initial condition $N(t_0) = N_0$, that is there are N_0 atoms at time t_0.

Proof: We first rearrange the equation in the above definition to see that

$$\frac{N(t+\delta t) - N(t)}{\delta t} = -kN.$$

Taking the limit $\delta t \to 0$, we obtain precisely the definition of the derivative at the start of §1.1.1. In other words, we are presented with this separable ODE, with initial condition $N(t_0) = N_0$:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN.$$

We solve this using the method discussed in §2.2. Doing exactly this, we obtain

$$\int \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t}\ \mathrm{d}t = \int -k\ \mathrm{d}t \qquad \Rightarrow \qquad \log\big(|N|\big) = -kt + C \qquad \Rightarrow \qquad |N| = e^{-kt}e^{C}.$$

Since $N > 0$, we replace $|N|$ with N. Furthermore, define $A \coloneqq e^C$ so the general solution reads

$$N(t) = Ae^{-kt}.$$

Next, the initial condition $N(t_0) = N_0$ can be used to obtain the as-of-yet unknown variable:

$$N_0 = Ae^{-kt_0} \qquad \Rightarrow \qquad A = N_0 e^{kt_0}.$$

Finally, we substitute this into the general solution to get $N(t) = N_0 e^{k(t_0 - t)} = N_0 e^{-k(t-t_0)}$. □

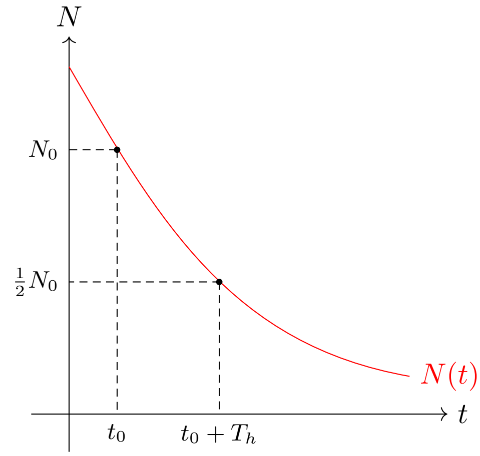

Figure 1: The plot of $N(t)$ for a general initial condition $N(t_0) = N_0$.

Definition The half-life is the time T_h after which half of the isotope has decayed.

Corollary *The half-life of a number of atoms with decay constant k, where $N(t_0) = N_0$, is*

$$T_h = \frac{1}{k}\log(2).$$

Proof: By the definition of half-life, we are interested in solving $N(t_0 + T_h) = \frac{1}{2}N_0$. Using the previous proposition, we can re-write $N(t_0 + T_h)$ and obtain the equation

$$N_0 e^{-k(t_0+T_h-t_0)} = \frac{1}{2}N_0 \qquad \Leftrightarrow \qquad N_0 e^{-kT_h} = \frac{1}{2}N_0.$$

Rearranging this for T_h is easy and it does indeed produce precisely what we want. □

3.1.2 Carbon Dating

There are three main isotopes of carbon: ^{12}C, ^{13}C, ^{14}C. The vast majority of carbon is made up of the first two; this is because the latter isotope is radioactive.

Note: The ^{14}C isotope decays with a half-life of 5730 years to form a nitrogen isotope ^{14}N.

Method – Carbon Dating: We can find the age of a fossil that has $i\%$ of its original ^{14}C isotope. Let t be today's date and suppose the fossil is T years old, meaning it fossilised at time $t_0 = t - T$. If x_0 is the number of atoms of ^{14}C at t_0, it now has $i\% \times x_0$ atoms.

(i) Use the previous proposition to see that $x(t) = x_0 e^{-kT}$.
(ii) Use the fact that we know $x(t) = i\% \times x_0$.
(iii) Equate the right-hand sides in Steps (i) and (ii) and solve for T.

3.2 Heating and Cooling

Definition Suppose we have an object that is hotter than the ambient temperature A. Then, Newton's Law of Cooling states that the object's temperature $\theta(t)$ satisfies

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - A),$$

where $\theta(t_0) = \theta_0$ is an initial condition and $k > 0$ is a positive constant.

Proposition *An object's temperature θ in an environment with temperature A satisfies*

$$\theta(t) = A + (\theta_0 - A)e^{-k(t-t_0)},$$

assuming the initial condition $\theta(t_0) = \theta_0$, that is the object has temperature θ_0 at time t_0.

Proof: Newton's Law of Cooling is a linear first-order ODE which can be written in the form

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} + k\theta = kA.$$

So, we use the integrating factor method in §2.4 to solve this. Indeed, the integrating factor is

$$R(t) = \exp\left(\int k \ \mathrm{d}t\right) = e^{kt}.$$

Multiplying the above ODE by this integrating factor produces

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\theta e^{kt}\right] = kAe^{kt}.$$

Integrating this, we obtain $\theta e^{kt} = Ae^{kt} + C$, which means the general solution of the ODE is

$$\theta(t) = A + Ce^{-kt}.$$

Using the initial condition, we obtain $C = (\theta_0 - A)e^{kt_0}$. Consequently, the particular solution is

$$\theta(t) = A + (\theta_0 - A)e^{kt_0}e^{-kt} = A + (\theta_0 - A)e^{-k(t-t_0)}. \qquad \square$$

Note: The temperature decays exponentially to the ambient temperature: $\lim_{t\to\infty} \theta(t) = A$.

3.3 Population Models

Let $p(t)$ be the population of a country at time t. The rate of change of a population in a country is equal to the rate at which people enter the country (e.g. births and immigration) minus the rate at which people leave the country (e.g. deaths and emigration). We can write this as

$$\frac{\mathrm{d}p}{\mathrm{d}t} = B(p,t) - D(p,t) + M(p,t),$$

where B represents the births, D represents the deaths and M is migration **into** the country.

3.3.1 The Malthusian Model

Definition The Malthusian model is this simple population model with migration $M = 0$:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = (b - d)p = \gamma p,$$

where $B(p,t) = bp(t)$ and $D(p,t) = dp(t)$ are proportional to the population and $\gamma = b - d$ is called the growth rate of the population (this can be positive or negative or zero).

Proposition *In the Malthusian model, a population at time t such that $p(t_0) = p_0$ satisfies*

$$p(t) = p_0 e^{\gamma(t-t_0)}.$$

Proof: The ODE in the definition of the Malthusian model is separable, so we can see that

$$\int \frac{1}{p}\ \mathrm{d}p = \int \gamma\ \mathrm{d}t \qquad \Rightarrow \qquad \log\big(|p|\big) = \gamma t + C.$$

Notice $p \geq 0$, so we replace $|p|$ with p. Taking the exponential and defining $A \coloneqq e^C$, we get

$$p(t) = Ae^{\gamma t}.$$

Substituting the initial conditions gives $p_0 = Ae^{\gamma t_0} \Leftrightarrow A = p_0 e^{-\gamma t_0}$. The particular solution is

$$p(t) = p_0 e^{-\gamma t_0} e^{\gamma t} = p_0 e^{\gamma(t-t_0)}. \qquad \square$$

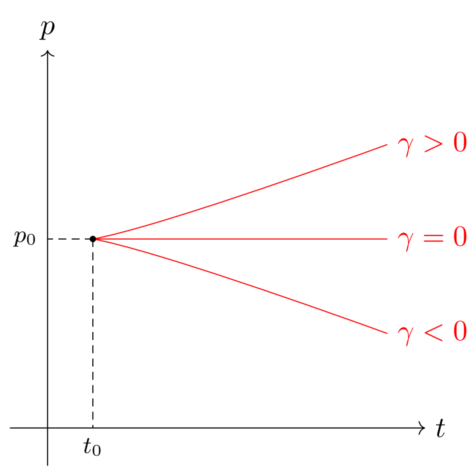

Figure 2: The plot of the Malthusian model's $p(t)$ for a general initial condition $p(t_0) = p_0$.

Note: The Malthusian model predicts either that (i) the population grows without bound ($\gamma > 0$) or (ii) the population decays to extinction ($\gamma < 0$). These are fairly unrealistic.

3.3.2 The Logistic Model

Definition A limiting population is a positive value $p_\infty > 0$ such that the following occur:

(i) The population p grows ($\gamma > 0$) when $p < p_\infty$.

(ii) The population p decays ($\gamma < 0$) when $p > p_\infty$.

(iii) The population p is unchanged ($\gamma = 0$) when $p = p_\infty$.

Definition The logistic model is this population model, with γ depending linearly on p:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \mu p\left(1 - \frac{p}{p_\infty}\right),$$

where $\mu > 0$ is a positive growth rate in the limit of a small population ($p \to 0$).

Note: This is the same form $\frac{\mathrm{d}p}{\mathrm{d}t} = \gamma p$ as the Malthusian model, but with $\gamma = \mu\left(1 - \frac{p}{p_\infty}\right)$.

Proposition *In the logistic model, a population at time t such that $p(t_0) = p_0$ satisfies*

$$p(t) = \frac{p_\infty p_0}{(p_\infty - p_0)e^{-\mu(t-t_0)} + p_0}.$$

Proof: The ODE in the definition of the logistic model is a Bernoulli equation, so the method in §2.4.1 can be used to solve it. Alternatively, we can recognise that it is a (non-linear) separable ODE; this is probably an easier route to take. Indeed, the logistic model is equivalent to

$$\begin{aligned}
&\int \frac{1}{p\left(1 - \frac{p}{p_\infty}\right)}\ \mathrm{d}p = \int \mu\ \mathrm{d}t + C \\
\Rightarrow\quad &\int \frac{p_\infty}{p(p_\infty - p)}\ \mathrm{d}p = \mu t + C \\
\Rightarrow\quad &\int \left(\frac{1}{p} + \frac{1}{p_\infty - p}\right)\ \mathrm{d}p = \mu t + C \\
\Rightarrow\quad &\log\big(|p|\big) - \log\big(|p_\infty - p|\big) = \mu t + C.
\end{aligned}$$

We ditch the absolute values, set $A \coloneqq \pm e^C$ and take the exponential of the above line to get

$$\frac{p}{p_\infty - p} = Ae^{\mu t}.$$

Substituting the initial conditions tells us that

$$\frac{p_0}{p_\infty - p_0} = Ae^{\mu t_0} \qquad \Leftrightarrow \qquad A = \frac{p_0}{p_\infty - p_0}e^{-\mu t_0}.$$

It remains to substitute this back into the (implicit) general solution above and rearrange:

$$\begin{aligned}
&& \frac{p}{p_\infty - p} &= \frac{p_0}{p_\infty - p_0} e^{\mu(t-t_0)} \\
\Rightarrow && p &= \frac{p_0}{p_\infty - p_0}(p_\infty - p) e^{\mu(t-t_0)} \\
\Rightarrow && (p_\infty - p_0)p + p_0 p e^{\mu(t-t_0)} &= p_0 p_\infty e^{\mu(t-t_0)} \\
\Rightarrow && p\left((p_\infty - p_0) + p_0 e^{\mu(t-t_0)}\right) &= p_0 p_\infty e^{\mu(t-t_0)} \\
\Rightarrow && p(t) &= \frac{p_\infty p_0 e^{\mu(t-t_0)}}{(p_\infty - p_0) + p_0 e^{\mu(t-t_0)}} \\
&& &= \frac{p_\infty p_0}{(p_\infty - p_0)e^{-\mu(t-t_0)} + p_0}.
\end{aligned}$$

□

Note: The limiting population truly does what it says on the tin, meaning $\lim_{t\to\infty} p(t) = p_\infty$.

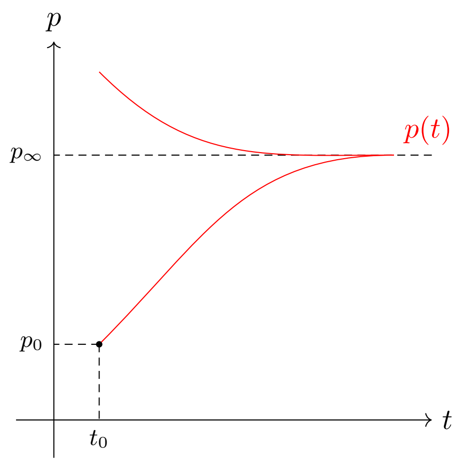

Figure 3: The plot of the logistic model's $p(t)$ for a general initial condition $p(t_0) = p_0$.

3.4 Particle Dynamics

The motion of a body (or particle) can be described by a simple ODE which relates the rate of change of its velocity to the forces acting upon it.

Definition Suppose we have a body in motion. Then, Newton's Second Law of Motion says that the rate of change of its momentum (mass times velocity) is precisely the sum of the forces acting upon it. This is expressed as the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[mv\right] = F,$$

where F is the sum of all applied forces.

Note: If the body's mass m is constant, Newton's Second Law of Motion reads $m\frac{\mathrm{d}v}{\mathrm{d}t} = F$, which relates the rate of change of its velocity (acceleration) to the forces acting upon it.

Lemma *A particle dropped from some height, subject to gravity only, has velocity given by*

$$v(t) = v_0 - gt$$

assuming the initial condition $v(0) = v_0$*, where* $g = 9.8\ ms^{-2}$ *is the gravitational strength.*

Proof: Here, the only force acting on the particle is gravity. Under the convention that up is positive and down is negative, the particle moves in the negative direction. Hence, the force acting upon the particle is $F = F_g = -mg$. By Newton's Second Law of Motion, we obtain

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg.$$

We can immediately integrate this to see that $v(t) = -gt + C$. Using the initial condition, we conclude that $C = v_0$; this is precisely the particular solution in the statement of the result. □

Definition A simple model of air resistance assumes that the force exerted by air resistance F_r is proportional to the velocity of the body in the opposite direction. In other words,

$$F_r = -kv,$$

where the positive number $k > 0$ is called the drag coefficient.

Note: The assumption F_r is directly proportional to negative velocity is good for slow-moving bodies. It is best to assume a so-called *quadratic model* for fast-moving bodies.

Proposition *A particle dropped from rest, subject to gravity and air resistance, has velocity*

$$v(t) = \frac{mg}{k}\left(e^{-(k/m)t} - 1\right).$$

Proof: Firstly, note that rest implies the initial velocity is zero, that is $v(0) = 0$; this will come in handy later. For now, we apply Newton's Second Law of Motion once again to obtain

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg - kv \qquad \Leftrightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}t} + Kv = -g,$$

where $K := k/m$. This is a linear first-order ODE and thus we can solve it using the integrating factor method from §2.4. Indeed, notice that the integrating factor here is

$$R(t) = \exp\left(\int K\ \mathrm{d}t\right) = e^{Kt}.$$

Hence, the above ODE can be re-written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{Kt}v\right] = -ge^{Kt}.$$

Integrating this and rearranging produces the general solution of the ODE, namely

$$e^{Kt}v = -\frac{g}{K}e^{Kt} + C \qquad \Rightarrow \qquad v(t) = -\frac{g}{K} + Ce^{-Kt}.$$

Applying the initial condition $v(0) = 0$, we get $C = g/K$. Therefore, the particular solution is

$$v(t) = -\frac{g}{K} + \frac{g}{K}e^{-Kt} = \frac{mg}{k}\left(e^{-(k/m)t} - 1\right). \qquad \square$$

Note: The velocity $v(t)$ above is dependent on the mass of the particle; this is expected.

Remark If we want to form a prediction at short times (i.e. $t \to 0$), then we can use the Taylor expansion of exp about $t = 0$ in the equation for $v(t)$ above. Indeed, we see that

$$v(t) \approx \frac{mg}{k}\left(1 - \frac{kt}{m} - 1\right) = -gt,$$

if we consider only the first two largest leading-order terms. This recovers the prediction of a body in free-fall (no air resistance) we got in the previous lemma (remembering that $v_0 = 0$).

Definition The terminal velocity of a falling body is the speed v_T at which drag and gravity are in balance. If $v > v_T$, the drag will be weaker than gravity and so the velocity will decrease back towards v_T. On the other hand, if $v < v_T$, the drag will be stronger than gravity and so the velocity will increase back towards v_T.

Corollary *For a particle dropped from rest, subject to gravity and air resistance, we have*

$$v_T = \frac{mg}{k}.$$

Proof: Use the velocity equation from the previous proposition and take the limit as $t \to \infty$. So,

$$\lim_{t\to\infty} v(t) = -\frac{mg}{k}.$$

But in the definition of terminal velocity, v_T is the **speed**, so taking the absolute value of the above (i.e. removing reference to direction) gives us precisely what we expect. $\square$

3.5 Economics and Finance

Financial systems provide another context where interactions between dependent variables lead to rich (pun not intended) dynamical systems. The main example here is that of a pension fund; this contains an amount of money with some interest rate, from which funds are withdrawn.

Definition A simple model for the balance of a bank account $B(t)$ at time t with interest rate r per annum (per year) and withdrawal rate W is

$$\frac{\mathrm{d}B}{\mathrm{d}t} = rB - W.$$

Proposition *Let a bank account has balance $B(t)$ at time t, where the amount is in British pound sterling and time is measured in years, with initial condition $B(0) = B_0$. If r is the interest rate per annum and W is the rate at which funds are removed from the account, then the account will be empty at time*

$$t = \frac{1}{r} \log\left(\frac{W}{W - rB_0}\right).$$

Proof: We use the first-order ODE in the above definition and solve it with an integrating factor. First, rearrange the ODE to get it in the form from which we find the integrating factor:

$$\frac{\mathrm{d}B}{\mathrm{d}t} - rB = -W \qquad \Rightarrow \qquad R(t) = \exp\left(\int -r \ \mathrm{d}t\right) = e^{-rt}.$$

Hence, we see that the ODE can be re-written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-rt}B\right] = -We^{-rt}.$$

Integrating this and rearranging gives us the general solution for the balance, that is

$$e^{-rt}B = \frac{W}{r}e^{-rt} + C \qquad \Rightarrow \qquad B(t) = \frac{W}{r} + Ce^{rt}.$$

The initial condition $B(0) = B_0$ implies that $C = B_0 - W/r$. The particular solution is therefore

$$B(t) = \frac{W}{r} + \left(B_0 - \frac{W}{r}\right)e^{rt}.$$

Finally, we must determine when the fund runs out, that is $B(t) = 0$. This is to say that

$$\begin{aligned}
& \frac{W}{r} + \left(B_0 - \frac{W}{r}\right)e^{rt} = 0 \\
\Rightarrow \qquad & \frac{W}{r} = \frac{W - rB_0}{r}e^{rt} \\
\Rightarrow \qquad & \frac{W}{W - rB_0} = e^{rt} \\
\Rightarrow \qquad & \log\left(\frac{W}{W - rB_0}\right) = rt \\
\Rightarrow \qquad & t = \frac{1}{r}\log\left(\frac{W}{W - rB_0}\right).
\end{aligned}$$

□

3.6 Fluid Dynamics

Fluid dynamics presents another opportunity to model physical phenomena with differential equations. Here, we consider a simple problem concerning the emptying of a draining container.

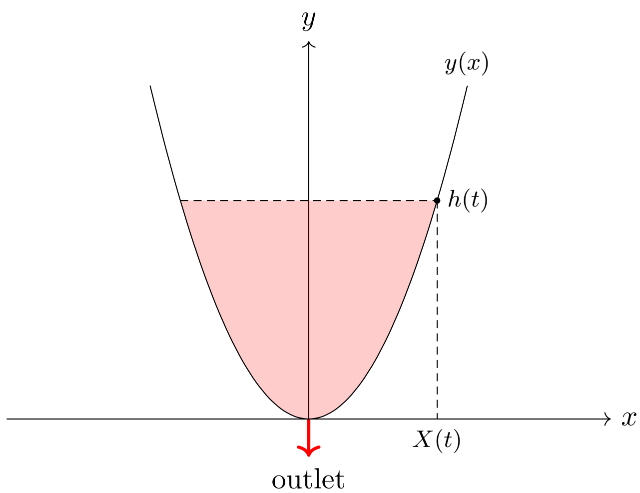

Figure 4: Draining a two-dimensional vessel $y = y(x)$ containing a fluid.

Definition The velocity $u(t)$ of the fluid in the outlet satisfies Bernoulli's Law, that is

$$\frac{1}{2}\rho u^2 = \rho g h,$$

where $h(t)$ is the height of the fluid in the two-dimensional vessel, $g = 9.8\ \mathrm{ms}^{-2}$ is the gravitational strength and ρ is the density of the fluid at all points in the fluid.

Note: By rearranging Bernoulli's Law, we see that the velocity in the outlet is $u(t) = \sqrt{2gh}$.

Proposition *Suppose $y = ax^n$ is the shape of a two-dimensional vessel, where $a, n > 0$ are positive constants. If the fluid has initial height $h(0) = h_0$, then the height satisfies*

$$h(t) = \left(h_0^{\frac{n+2}{2n}} - \frac{n+2}{2n}kt\right)^{\frac{2n}{n+2}}.$$

Proof: We must develop an ODE that describes the evolution of the height $h(t)$ of the fluid. First, suppose that the total volume of fluid in the vessel at time t is given by $V(t)$. By Bernoulli's Law, the amount of fluid leaving the outlet per unit of time is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -Su = -S\sqrt{2gh},$$

where S is the cross-sectional area of the outlet nozzle.

We must use the above differential equation to derive an ODE relating $\mathrm{d}h/\mathrm{d}t$ and h, which we can then solve. To do this, we find a relation between $h(t)$ and $V(t)$. Per Figure 4, the volume is essentially the area $A(t)$ between the surface of the fluid $h(t)$ and the boundary of the container $y(x)$. We can write this as the area of the rectangle with width $2X(t)$ and height $h(t)$ minus the area under the curve $y = y(x)$. Mathematically, this is

$$\begin{aligned} A(t) &= 2h(t)X(t) - \int_{-X(t)}^{X(t)} y(x)\ \mathrm{d}x \\ &= 2h(t)X(t) - 2\int_{0}^{X(t)} y(x)\ \mathrm{d}x, \end{aligned}$$

since $y(x)$ is symmetric about the y-axis. Using the expression $y(x) = ax^n$ we have, we see that

$$\begin{aligned} A(t) &= 2h(t)X(t) - 2\int_{0}^{X(t)} ax^n\ \mathrm{d}x \\ &= 2h(t)X(t) - 2a\left[\frac{x^{n+1}}{n+1}\right]_0^{X(t)} \\ &= 2h(t)X(t) - \frac{2a}{n+1}X(t)^{n+1}. \end{aligned}$$

We know that $y\big(X(t)\big) = h(t)$, by definition of $X(t)$. In other words, this says that $aX(t)^n = h(t)$, so we can eliminate $X(t)$ in favour of $h(t)$ in the formula for the area above and simplify:

$$\begin{aligned} A(t) &= 2h(t)\frac{h(t)^{1/n}}{a^{1/n}} - \frac{2a}{n+1}\frac{h(t)^{(n+1)/n}}{a^{(n+1)/n}} \\ &= \left(\frac{2}{a^{1/n}} - \frac{2}{(n+1)a^{1/n}}\right)h(t)^{(n+1)/n} \\ &= \frac{2n}{(n+1)a^{1/n}}h(t)^{(n+1)/n}. \end{aligned}$$

The formula we've known since pre-university for volume is "*cross-sectional area* $\times$ *length*". If L is the length of the vessel (in the direction into the page when looking at Figure 4), we see that

$$V(t) = A(t)L = \frac{2nL}{(n+1)a^{1/n}}h(t)^{(n+1)/n}.$$

If we substitute this into the ODE defined at the start of the proof, we obtain

$$\frac{2nL}{(n+1)a^{1/n}}\frac{\mathrm{d}}{\mathrm{d}t}\left[h^{(n+1)/n}\right] = -S\sqrt{2g}h^{1/2}.$$

Now, the Chain Rule applies to the derivative on the left-hand side, and we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[h^{(n+1)/n}\right] = \frac{n+1}{n}h^{1/n}\frac{\mathrm{d}h}{\mathrm{d}t}.$$

We can substitute this into the ODE and bring all the factors of $h(t)$ to the right, giving us

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -kh^{\frac{1}{2}-\frac{1}{n}}, \qquad \text{where } k := \sqrt{\frac{g}{2}}\frac{Sa^{1/n}}{L}.$$

Note that k defined above is a constant that lumps together all the constants in the problem thus far. The above ODE gives us what we want: an independent equation describing the evolution of the height $h(t)$ for any given vessel of the shape $y(x) = ax^n$. Moreover, this ODE is separable, so we can use the method of §2.2 to solve it: for a general solution:

$$\begin{aligned}
\frac{\mathrm{d}h}{\mathrm{d}t} = -kh^{\frac{1}{2}-\frac{1}{n}} \Leftrightarrow \quad & h^{\frac{1}{n}-\frac{1}{2}}\frac{\mathrm{d}h}{\mathrm{d}t} = -kh^{\frac{1}{2}-\frac{1}{n}} \\
\Rightarrow & \int h^{\frac{1}{n}-\frac{1}{2}}\frac{\mathrm{d}h}{\mathrm{d}t}\ \mathrm{d}t = \int -k\ \mathrm{d}t \\
\Rightarrow \quad & \int h^{\frac{1}{n}-\frac{1}{2}}\ \mathrm{d}h = -kt + C \\
\Rightarrow \quad & \frac{1}{\frac{1}{n}+\frac{1}{2}} h^{\frac{1}{n}+\frac{1}{2}} = -kt + C \\
\Leftrightarrow \quad & \frac{2n}{n+2} h^{\frac{2n}{n+2}} = -kt + C \\
\Rightarrow \quad & h(t) = \left(C - \frac{n+2}{2n}kt\right)^{\frac{2n}{n+2}}.
\end{aligned}$$

Using the initial condition $h(0) = h_0$ to find C, we obtain the intended particular solution, i.e.

$$h(t) = \left(h_0^{\frac{n+2}{2n}} - \frac{n+2}{2n}kt\right)^{\frac{2n}{n+2}}. \qquad \square$$

Note: We can find n such that $h(t)$ decreases equal heights in equal times; this means $h(t)$ must be linear in time. Now, this occurs when $2n/(n+2) = 1$ from which we get $n = 2$.

4 Linear Second-Order ODEs

The general problem of solving a linear second-order ordinary differential equation takes the form

$$y'' + f(x)y' + g(x)y = h(x), \qquad \text{with } y(x_0) = y_0.$$

4.1 Homogeneous Linear Second-Order ODEs

Definition Consider the above linear second-order ODE.

(i) It is called homogeneous if $h(x) \equiv 0$.

(ii) It is called inhomogeneous if $h(x) \not\equiv 0$.

Note: The symbol $\equiv$ means "*identically equal to*", so we are saying $h(x)$ is zero for **all** x.

Theorem (Superposition Principle) *Consider the homogeneous linear second-order ODE*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = 0.$$

If $y_1(x)$ and $y_2(x)$ are two solutions, then their superposition *$y(x) = Ay_1(x) + By_2(x)$ is also a solution, for* **any** *arbitrary constants A and B.*

Proof: We need only substitute $y = Ay_1 + By_2$ into the ODE and check we get zero. Indeed,

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big[Ay_1 + By_2\Big] + f(x)\frac{\mathrm{d}}{\mathrm{d}x}\Big[Ay_1 + By_2\Big] + g(x)(Ay_1 + By_2) \\
&= A\frac{\mathrm{d}^2y_1}{\mathrm{d}x^2} + B\frac{\mathrm{d}^2y_2}{\mathrm{d}x^2} + f(x)A\frac{\mathrm{d}y_1}{\mathrm{d}x} + f(x)B\frac{\mathrm{d}y_2}{\mathrm{d}x} + g(x)Ay_1 + g(x)By_2 \\
&= A\left(\frac{\mathrm{d}^2y_1}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y_1}{\mathrm{d}x} + g(x)y_1\right) + B\left(\frac{\mathrm{d}^2y_2}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y_2}{\mathrm{d}x} + g(x)y_2\right) \\
&= 0,
\end{aligned}$$

where each of the brackets in the second-to-last line is zero since y_1 and y_2 are solutions. $\square$

Note: The Superposition Principle **fails** for inhomogeneous ODEs and non-linear ODEs.

A general solution to a second-order ODE will have two arbitrary constants. Therefore, if we can find an expression for a solution with two constants, then we have ourselves a general solution; this is precisely what is provided by the superposition of two solutions! Therefore, to solve a homogeneous linear second-order ODE, we just need to find two *independent* solutions (we will define this word next).

Definition Two functions $y_1(x)$ and $y_2(x)$ are linearly independent if they are **not** related via multiplication by a constant, that is $y_1(x) \neq ky_2(x)$ for all x and some $k \in \mathbb{R}$. Otherwise, they are called linearly dependent.

Method – Showing Linear Independence: To show that two functions $y_1(x)$ and $y_2(x)$ are linearly independent, it suffices to look at their quotient, that is $y_1(x)/y_2(x)$. This fraction will **not** cancel to a constant if and only if the functions are linearly independent.

Remark Why do we require two solutions of a second-order ODE be linearly independent when we form the general solution from their superposition? Well, this is because if they are linearly dependent, we can just lump all the constants together into one constant; remember that we require **two** arbitrary constants in the general solution. Indeed, suppose that $y_1(x)$ and $y_2(x)$ are two linearly dependent solutions, meaning there exists constant k such that $y_2(x) = ky_1(x)$. Then, their superposition is

$$Ay_1(x) + By_2(x) = Ay_1(x) + Bky_1(x) = (A + Bk)y_1(x) =: Cy_1(x),$$

where $C := A + Bk$ is a constant. We only have **one** arbitrary constant here, which is not good!

4.1.1 The Case of Constant Coefficients

We consider a homogeneous linear second-order ODE where the coefficients a and b are constant:

$$y'' + ay' + by = 0.$$

Note: If the coefficient of y'' is not one, we can divide through so that it becomes one.

We propose a "*trial solution*" to the above ODE of the form $y(x) = e^{\lambda x}$, where λ is an unknown constant. Such a function works because its derivatives $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ retain the exponential factor $e^{\lambda x}$, which can be made to cancel on suitable choice of λ.

Definition The characteristic equation of the above ODE is the quadratic $\lambda^2 + a\lambda + b = 0$.

Theorem *Consider a homogeneous linear second-order ODE with constant coefficients.*

(i) *If the characteristic equation has two real roots λ_1 and λ_2, the general solution is*

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

(ii) *If the characteristic equation has one real repeated root λ, the general solution is*

$$y(x) = (A + Bx)e^{\lambda x}.$$

(iii) *If the characteristic equation has two complex roots $p \pm iq$, the general solution is*

$$y(x) = e^{px}\left(A\cos(qx) + B\sin(qx)\right).$$

Proof: Suppose we have a trial solution $y(x) = e^{\lambda x}$. Substituting y'', y', y into the ODE gives us

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = (\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Because $e^{\lambda x} > 0$ is always positive, we see that the above equation is satisfied precisely when $\lambda^2 + a\lambda + b = 0$; we can say that $y(x) = e^{\lambda x}$ is a solution when λ is a root of the characteristic equation. For the characteristic equation, the discriminant is the expression $\Delta := a^2 - 4b$.

(i) If the characteristic equation has two real roots λ_1 and λ_2, this means that $\Delta > 0$. Also,

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

Consequently, the functions $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ are solutions of the ODE. We see also that they are independent and thus their superposition is a general solution; their superposition is precisely

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

(ii) If the characteristic equation has one real repeated root λ, this means that $\Delta = 0$. Also,

$$\lambda = -\frac{a}{2}.$$

Thus, the function $y_1(x) = e^{\lambda x}$ is a solution of the ODE; we need a second independent solution. The trick is to define y_2 as y_1 multiplied by a non-constant (so that $y_1(x)/y_2(x)$ is **not** a constant). One such example would be to consider $y_2(x) = xe^{\lambda x}$. Before we proceed, we check that this is indeed a solution by substituting $y = y_2$ into the ODE:

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a\frac{\mathrm{d}y}{\mathrm{d}x} + by &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[xe^{\lambda x}\right] + a\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^{\lambda x}\right] + b(xe^{\lambda x}) \\ &= \lambda(\lambda x + 2)e^{\lambda x} + a(\lambda x + 1)e^{\lambda x} + bxe^{\lambda x} \\ &= \Big((\lambda^2 + a\lambda + b)x + 2\lambda + a\Big)\, e^{\lambda x} \\ &= 0,\end{aligned}$$

because $\lambda = -a/2$ which means that both $\lambda^2 + a\lambda + b = 0$ and $2\lambda + a = 0$. Thus, y_2 is also a solution of the ODE, and we know it is independent from y_1; we can form the superposition once again to obtain a general solution which, in this case, is

$$y(x) = (A + Bx)e^{\lambda x}.$$

(iii) If the characteristic equation has two complex roots $p \pm iq$, this means that $\Delta < 0$. Also,

$$p = -\frac{a}{2} \qquad \text{and} \qquad q = \frac{\sqrt{4b - a^2}}{2}.$$

Similar to the real case, the functions $y_1(x) = e^{(p+iq)x}$ and $y_2(x) = e^{(p-iq)x}$ are solutions of the ODE. Again then, their superposition provides a general solution and the constants are possibly complex. We can use Euler's formula $\exp(ix) = \cos(x) + i\sin(x)$ to write

$$y(x) = Ce^{(p+iq)x} + De^{(p-iq)x}$$

$$
\begin{aligned}
&= e^{px}\left(Ce^{iqx} + De^{-iqx}\right) \\
&= e^{px}\left(C\left(\cos(qx) + i\sin(qx)\right) + D\left(\cos(qx) - i\sin(qx)\right)\right) \\
&= e^{px}\left((C+D)\cos(qx) + i(C-D)\sin(qx)\right) \\
&= e^{px}\left(A\cos(qx) + B\sin(qx)\right),
\end{aligned}
$$

where we define new constants $A \coloneqq C + D$ and $B \coloneqq i(C - D)$. □

Method – Solving Homogeneous Linear Constant-Coefficient Second-Order ODEs: For a homogeneous linear second-order ODE whose coefficients are constant, we can apply the previous theorem to give us a general solution with relatively-little hassle.

(i) If the y''-coefficient is one, move to Step (ii). If not, divide by the current coefficient.
(ii) Find the roots of the characteristic equation.
(iii) Substitute them into the relevant general solution from the above theorem.

4.2 Inhomogeneous Linear Second-Order ODEs

We consider an inhomogeneous second-order ODE with arbitrary coefficients $f(x)$ and $g(x)$:

$$y'' + f(x)y' + g(x)y = h(x), \qquad \text{with } h(x) \not\equiv 0.$$

Definition Consider the above inhomogeneous second-order ODE.

(i) Any solution to $y'' + f(x)y' + g(x)y = h(x)$ is a particular integral, denoted $y_P(x)$.
(ii) A solution to $y'' + f(x)y' + g(x)y = 0$ is a complementary function, denoted $y_C(x)$.

Proposition *Consider the inhomogeneous second-order ODE*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x).$$

If $y_C(x)$ is a complementary function (solution to the homogeneous-version of the ODE) and $y_P(x)$ is a particular integral (solution to the above ODE), then it follows that their sum $y(x) = y_C(x) + y_P(x)$ is a solution of the above ODE also.

Proof: It remains to substitute $y = y_C + y_P$ into the inhomogeneous ODE:

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[y_C + y_P\right] + f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[y_C + y_P\right] + g(x)(y_C + y_P) \\
&= \left(\frac{\mathrm{d}^2y_C}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y_C}{\mathrm{d}x} + g(x)y_C\right) + \left(\frac{\mathrm{d}^2y_P}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y_P}{\mathrm{d}x} + g(x)y_P\right) \\
&= h(x),
\end{aligned}
$$

noting that $y_C(x)$ is a solution to the homogeneous-version, meaning the first bracket is zero, and $y_P(x)$ is a solution to the ODE itself, meaning the second bracket is $h(x)$. □

Note: The particular integral $y_P(x)$ is **not** at all unique; any solution to the ODE will do.

Method – Solving Inhomogeneous Second-Order ODEs: Suppose we have an ODE of the form written out in the above proposition.

(i) Find a general solution $y_C(x)$ to the homogeneous-version using the previous method.
(ii) Determine **any** solution $y_P(x)$ to the given inhomogeneous ODE.
(iii) Add together the two solutions from Steps (i) and (ii) to get a general solution.

4.2.1 Method of Undetermined Coefficients: Constructing Particular Integrals

The above method is good but, in practice, it may be difficult to do Step (ii), that is find a solution to the inhomogeneous ODE we are given. We next develop an approach for constructing a particular integral when the inhomogeneous ODE has constant coefficients, i.e. it's of the form

$$y'' + ay' + by = h(x).$$

Note: We can again divide through to ensure the coefficient of y'' is one. Of course, this will change $h(x)$ but only up to multiplicative constant so it isn't too bad.

We again propose a "*trial solution*" to the above ODE of a form which looks a bit like $h(x)$ but includes some constants. We then substitute this trial solution in and determine what the constants must be to ensure it satisfies the ODE.

Method – Choosing a Trial Solution: We present a number of cases for trial solutions of the inhomogeneous second-order ODE with constant coefficients, depending on $h(x)$.

(i) If $h(x)$ is a polynomial of degree n, use $y(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.
(ii) If $h(x)$ is an exponential, e.g. e^{kx}, use $y(x) = ae^{kx}$.
(iii) If $h(x)$ contains sinusoids, e.g. $\cos(\omega x)$ or $\sin(\omega x)$, use $y(x) = a\cos(\omega x) + b\sin(\omega x)$.
(iv) If $h(x)$ is a product of (ii) and (iii), use $y(x) = e^{kx}\big(a\cos(\omega x) + b\sin(\omega x)\big)$.

Remark If $h(x)$ is a polynomial of degree n, the trial solution should include **every** power of x up to x^n. For example, if $h(x) = x^3 + 2$, we should try $y(x) = ax^3 + bx^2 + cx + d$. A similar thing holds if $h(x)$ contains one trigonometric function and not the other; the trial solution should include **both**. For example, if $h(x) = \cos(2\pi x)$, we should try $y(x) = a\cos(2\pi x) + b\sin(2\pi x)$.

Note: If the trial solution is (part of) the complementary function, things **won't** work how we want them to. However, there is an easy fix: multiply the first trial solution by x.

We also note the following extra rules:

- If $h(x)$ is a sum of functions, the trial solution should be the sum of the trials of each of the functions. For example, if $h(x) = e^x + \sin(x)$, we should try $y(x) = ae^x + b\cos(x) + c\sin(x)$.
- If multiplying the trial solution by x again gives us (part of) the complementary function, just multiply again; we ultimately end up with multiplying the first trial solution by x^2.

4.3 Reduction of Order Method

We revert back to general coefficients $f(x)$ and $g(x)$ of a linear second-order ODE and discuss a method used to solve it, provided we know a solution of the **homogeneous** ODE a priori.

Lemma *Consider the inhomogeneous second-order ODE*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x)$$

and let $y_1(x)$ be a solution to the homogeneous-version. Then, we re-write the above as

$$y_1(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \big(2y_1'(x) + f(x)y_1(x)\big)\frac{\mathrm{d}u}{\mathrm{d}x} = h(x),$$

where $u(x)$ is a new dependent variable defined by setting $y(x) = y_1(x)u(x)$.

Proof: We must substitute $y = y_1u$ into the ODE and simplify things where possible. To begin, we shall write the first and second derivatives of this new function using the Product Rule:

$$y'(x) = y_1(x)u'(x) + y_1'(x)u(x) \qquad \text{and} \qquad y_1''(x) = y_1(x)u''(x) + 2y_1'(x)u'(x) + y_1''(x)u(x).$$

Performing the substitution (and omitting some of the "(x)" to ease notation) leaves us with

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big[y_1u\Big] + f(x)\frac{\mathrm{d}}{\mathrm{d}x}\Big[y_1u\Big] + g(x)y_1u \\ &= (y_1u'' + 2y_1'u' + y_1''u) + f(x)(y_1u' + y_1'u) + g(x)(y_1u) \\ &= y_1u'' + (2y_1' + f(x)y_1)u' + (y_1'' + f(x)y_1' + g(x)y_1)u \\ &= y_1u'' + (2y_1' + f(x)y_1)u',\end{aligned}$$

since $y_1(x)$ is assumed to be a solution of the homogeneous-version, meaning the end bracket in the second-to-last line is zero. This is the left-hand side of the ODE, the right of which is $h(x)$. Thus, we obtain the new ODE as written in the statement of the lemma. □

Note: The ODE obtained above is a first-order linear ODE in the variable $v := u'$, that is

$$y_1(x)\frac{\mathrm{d}v}{\mathrm{d}x} + \big(2y_1'(x) + f(x)y_1(x)\big)\,v(x) = h(x).$$

Method – Solving ODEs via Reduction of Order: Suppose we have an inhomogeneous second-order ODE and know a solution to the homogeneous version.

(i) Transform the ODE to a first-order linear ODE as in the above lemma and note.
(ii) Solve the ODE from Step (i) via the integrating factor method for $v(x) = u'(x)$.
(iii) Integrate the solution to obtain $u(x)$.
(iv) Use the expression for $u(x)$ in Step (iii) to get the general solution $y(x) = y_1(x)u(x)$.

But why does making the substitution $y = y_1 u$ reduce the order of an ODE? To get an idea of this, recall that the general solution of an inhomogeneous linear ODE has the form

$$y(x) = Ay_1(x) + By_2(x) + y_P(x),$$

where y_1 and y_2 are linearly independent solutions of the homogeneous-version (and thus their superposition forms the general solution of the homogeneous-version, i.e. the complementary function). If we divide the above equation by y_1, we see that

$$\frac{y(x)}{y_1(x)} = A + B\frac{y_2(x)}{y_1(x)} + \frac{y_P(x)}{y_1(x)}.$$

By defining $u \coloneqq y/y_1$, the above can be written

$$u(x) = A + Bu_2(x) + u_P(x)$$

for $u_2 \coloneqq y_2/y_1$ and $u_P \coloneqq y_P/y_1$. The reduction of order method produces an ODE in u', and

$$u'(x) = Bu_2'(x) + u_P'(x) \qquad \Leftrightarrow \qquad v(x) = Bv_2(x) + v_P(x),$$

when we define $v \coloneqq u'$, $v_2 \coloneqq u_2'$ and $v_P \coloneqq u_P'$. Clearly, it depends only on a single arbitrary constant B, so if it's a general solution to an ODE, said differential equation is first-order.

> **Note:** This works for **every** order, e.g. reduce linear third-order to linear second-order.

5 Two-Variable ODEs and the Phase Plane

In this section, we discuss two-dimensional systems of first-order ODEs. These systems involve **two** dependent variables, x and y say, which each depend on the same independent variable t, that is $x = x(t)$ and $y = y(t)$ are functions of time.

Definition A two-dimensional system of first-order ODEs is one of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x, y, t) \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = G(x, y, t).$$

Such a system has an initial condition which looks like $\big(x(t_0), y(t_0)\big) = (x_0, y_0)$.

Note: We restrict to homogeneous linear systems with constant coefficients, namely

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy. \tag{5.2}$$

The aim of the game is to determine $x(t)$ and $y(t)$. However, this can be difficult in general because the equations for x and y are *coupled*, meaning the solution $x(t)$ of the first ODE relies on knowing what $y(t)$ is; this is determined by the second ODE, which relies on knowing $x(t)$!

Remark Conveniently, we can re-write the system (5.2) of interest using matrix notation, that is

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.3}$$

which we can give even shorter notation for in the form of $\dot{\mathbf{x}} = \boldsymbol{A}\mathbf{x}$, where $\boldsymbol{A}$ is the 2×2 matrix and $\mathbf{x}$ is the vector on the right-hand side of (5.3).

Note: The dot notation $\dot{\mathbf{x}}$ means we differentiate the contents of the vector; some physicists and mathematicians use dot notion whenever they differentiate with respect to time t.

Definition A fixed point of the system $\boldsymbol{A}\mathbf{x} = \mathbf{0}$ is a constant solution, that is where $\dot{\mathbf{x}} = \mathbf{0}$.

Reminder: The determinant of a 2×2 matrix is the constant defined as

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

We call a matrix $\boldsymbol{A}$ invertible if there exists a matrix $\boldsymbol{A}$ such that $\boldsymbol{AB} = I = \boldsymbol{BA}$, where $\boldsymbol{I}$ is the so-called identity matrix; this has ones on the main diagonal and zeros elsewhere. The key result (for any $n \times n$ matrix) is that $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

Assuming that $\det(\boldsymbol{A}) \neq 0$ for the matrix $\boldsymbol{A}$ in (5.3), which we assume throughout the remainder of MATH1400, the unique fixed point that occurs in the examples we consider is $\mathbf{x} = \mathbf{0} = (0, 0)$.

5.1 Solving Two-Variable ODEs

Throughout this subsection, the system of ODEs we consider is that in (5.2), or rather (5.3).

Reminder: A matrix $\boldsymbol{A}$ has eigenvector $\mathbf{v}$ with corresponding eigenvalue λ when $\boldsymbol{A}\mathbf{v} = \lambda\mathbf{v}$.

Theorem *Consider the system of homogeneous linear first-order ODEs given in* (5.3).

(i) *If* $\boldsymbol{A}$ *has two real eigenvalues* λ_1 *and* λ_2 *with corresponding eigenvectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *respectively, then the general solution is*

$$\mathbf{x}(t) = A\mathbf{v}_1 e^{\lambda_1 t} + B\mathbf{v}_2 e^{\lambda_2 t}.$$

(ii) *If* $\boldsymbol{A}$ *has one real repeated eigenvalue* λ *with corresponding eigenvector* $\mathbf{v}$*, and* $\mathbf{w}$ *is any vector which satisfies* $(\boldsymbol{A} - \lambda\boldsymbol{I})\mathbf{w} = \mathbf{v}$*, then the general solution is*

$$\mathbf{x}(t) = A\mathbf{v}e^{\lambda t} + B(t\mathbf{v} + \mathbf{w})e^{\lambda t}.$$

(iii) *If* $\boldsymbol{A}$ *has two complex eigenvalues* $p \pm iq$ *with corresponding eigenvectors* $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$ *and* $\mathbf{v}_2 = \mathbf{c} + i\mathbf{d}$ *respectively, then the general solution is*

$$\mathbf{x}(t) = e^{pt}\left((A\mathbf{a} + B\mathbf{b})\cos(qt) + (B\mathbf{a} - A\mathbf{b})\sin(qt)\right).$$

Note: This result has very much the same flavour as the theorem we proved in §4.1.1.

Proof: From our experience with linear first-order ODEs, we try a trial solution of the form

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}, \qquad \text{where } \mathbf{v} = \begin{pmatrix} u \\ v \end{pmatrix},$$

for some constants u, v, λ. The idea is to determine them by substituting the above trial into the system. Indeed, doing so tells us that

$$\dot{\mathbf{x}} = \boldsymbol{A}\mathbf{x} \qquad \Leftrightarrow \qquad \lambda\mathbf{v}e^{\lambda t} = \boldsymbol{A}\mathbf{v}e^{\lambda t}.$$

Because $e^{\lambda t} > 0$, in particular it is non-zero, we can divide it out of the equation above, leaving us with defining equation of eigenvectors and eigenvalues: $\boldsymbol{A}\mathbf{v} = \lambda\mathbf{v}$. Thus, solutions of this system are determined by the eigendata of the matrix $\boldsymbol{A}$, namely u and v form the eigenvector and λ is the corresponding eigenvalue.

We can apply the Superposition Principle here (each of the two first-order ODEs is linear), so the general solution we expect is of the form $\mathbf{x}(t) = A\mathbf{x}_1(t) + B\mathbf{x}_2(t)$, where $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$ come from this eigenvector/eigenvalue analysis. Furthermore, recall that the eigenvalues are found by solving $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$, which for us is just this quadratic equation:

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0.$$

Let the discriminant of this quadratic be denoted $\Delta \coloneqq (a+d)^2 - 4(ad-bc) = \mathrm{tr}(A)^2 - 4\det(A)$.

(i) If we have two real eigenvalues λ_1 and λ_2, this means that $\Delta > 0$. As such, the discussion above immediately tells us that $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$. So, the Superposition Principle provides us with the general solution we expect:

$$\mathbf{x}(t) = A\mathbf{v}_1 e^{\lambda_1 t} + B\mathbf{v}_2 e^{\lambda_2 t}.$$

(ii) (**Non-examinable**) If we have one real repeated eigenvalue λ, this means that $\Delta = 0$; we will not encounter this case during this course and thus we will not provide the proof.

(iii) If we have two complex eigenvalues $p \pm iq$, this means that $\Delta > 0$. Similar to the real case, we have independent solutions $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$, from which we obtain the general solution via superposition. That said, we would prefer an expression for a solution that is explicitly real; this is because most applications of this theory involve terms, coefficients, initial conditions, parameters, etc. that are all real. To do this, we again apply Euler's formula. First, we use the following small result (which we prove).

> **Lemma** *If $\mathbf{v}_1$ and $\mathbf{v}_2$ are complex eigenvectors, then they are conjugate, i.e. $\mathbf{v}_1^* = \mathbf{v}_2$.*

Proof of the Lemma: By definition of an eigenvector, we know that $\boldsymbol{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$. Hence, taking the conjugate yields the equation $\boldsymbol{A}^*\mathbf{v}_1^* = \lambda_1^*\mathbf{v}_1^*$; we have used the basic fact that the conjugate of a product is the product of the conjugates. Because $\boldsymbol{A}$ is a **real** matrix, its conjugate does noting, i.e. $\boldsymbol{A}^* = \boldsymbol{A}$. Also, as $\lambda_1 = p + iq$ and $\lambda_2 = p - iq$, we immediately see that $\lambda_1^* = \lambda_2$. Therefore, we are really working with the equation $\boldsymbol{A}\mathbf{v}_1^* = \lambda_2\mathbf{v}_1^*$, but because $\mathbf{v}_2$ is the eigenvector with corresponding eigenvalue λ_2, we see that $\mathbf{v}_1^* = \mathbf{v}_2$. □

Continuing the proof of the theorem then, suppose we write the first eigenvector in terms of two real vectors, namely $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$. The above lemma tells us that $\mathbf{v}_2 = \mathbf{a} - i\mathbf{b}$. We can substitute these, along with $\lambda_1 = p + iq$ and $\lambda_2 = p - iq$ into the superposition of the two independent solutions mentioned at the start of this case:

$$\begin{aligned}
\mathbf{x}(t) &= C(\mathbf{a} + i\mathbf{b})e^{(p+iq)t} + B(\mathbf{a} - i\mathbf{b})e^{(p-iq)t} \\
&= e^{pt}\left(C(\mathbf{a} + i\mathbf{b})e^{iqt} + B(\mathbf{a} - i\mathbf{b})e^{-iqt}\right) \\
&= e^{pt}\left(C(\mathbf{a} + i\mathbf{b})\big(\cos(qt) + i\sin(qt)\big) + B(\mathbf{a} - i\mathbf{b})\big(\cos(qt) - i\sin(qt)\big)\right) \\
&= e^{pt}\left(\big((C+D)\mathbf{a} + i(C-D)\mathbf{b}\big)\cos(qt) + \big(i(C-D)\mathbf{a} - (C+D)\mathbf{b}\big)\sin(qt)\right) \\
&= e^{pt}\big((A\mathbf{a} + B\mathbf{b})\cos(qt) + (B\mathbf{a} - A\mathbf{b})\sin(qt)\big)
\end{aligned}$$

where we define new constants $A \coloneqq C + D$ and $B \coloneqq i(C - D)$. □

> **Method – Solving a System of Homogeneous Linear Constant-Coefficient ODEs:** Let's assume we are given a system of first-order ODEs of the form (5.2) that we wish to solve.
> (i) Write the system in matrix form (5.3).
> (ii) Find the eigenvalues λ_1 and λ_2 and eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the matrix $\boldsymbol{A}$.
> (iii) Substitute them into the relevant general solution from the above theorem.

5.2 Phase Portraits and Classifying Fixed Points

Definition For a two-dimensional system (5.2), the phase plane is the xy-plane on which we plot the solutions $\big(x(t), y(t)\big)$. The curve this pair traces as we vary t is a trajectory. We say its orientation is positive as t increases and negative as t decreases; this is indicated by an arrow. A diagram showing multiple trajectories **with arrows** is a phase portrait.

In §5.1, we learned that if a system is initially at a fixed point, then it stays at said fixed point for all future t. The natural question is this: what happens if we start **near** a fixed point; does the trajectory move towards it or away from it?

Definition We define the following for a fixed point $\mathbf{x}_*$ of a system of ODEs:

(i) It is stable if trajectories converge to it, that is $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_*$.

(ii) It is unstable if trajectories diverge to infinity, that is $\lim_{t\to\infty} \mathbf{x}(t) = \infty$.

(iii) It is neutrally stable if trajectories neither approach it or diverge to infinity.

Note: Imaging a ball in a valley; there is a stable fixed point at the bottom of the valley (because the ball will always roll back into the valley if it is close enough to the base) and, analogously, there is an unstable fixed point at the top of the valley.

Definition We also define the following for a fixed point of a system of ODEs:

(i) It is a node if all trajectories approach it in a definite direction for both $t \to \pm\infty$.

(ii) It is a saddle if some trajectories approach it as $t \to \infty$ and others as $t \to -\infty$.

(iii) It is a centre if all trajectories form closed curves that encircle it.

(iv) It is a spiral if individual trajectories 'orbit' it forever as $t \to \infty$.

We proceed to classify the form of six of the most fundamental types of fixed point. However, we exclude a few 'exotic' situations where the eigenvalues are equal (so-called *star nodes*) or when one eigenvalue is zero (so-called *non-isolated nodes*).

Theorem (Classification of Fixed Points) *Consider a two-dimensional system of ODEs in matrix notation with matrix $\boldsymbol{A}$, and denote the eigenvalues by λ_1 and λ_2.*

(i) *If $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 < 0$ and $\lambda_2 < 0$, the fixed point is a* **stable node**.

(ii) *If $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 > 0$ and $\lambda_2 > 0$, the fixed point is an* **unstable node**.

(iii) *If $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 > 0$ and $\lambda_2 < 0$, the fixed point is a* **saddle**.

(iv) *If $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_1) = 0$ and $\mathrm{Re}(\lambda_2) = 0$, the fixed point is a* **centre**.

(v) *If $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_1) < 0$ and $\mathrm{Re}(\lambda_2) < 0$, the fixed point is a* **stable spiral**.

(vi) *If $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_1) > 0$ and $\mathrm{Re}(\lambda_2) > 0$, the fixed point is an* **unstable spiral**.

Proof: We proceed on a case-by-case basis.

(i) Since the eigenvalues are real, this falls under Case (i) of the previous theorem, meaning the general solution is $\mathbf{x}(t) = A\mathbf{v}_1 e^{\lambda_1 t} + B\mathbf{v}_2 e^{\lambda_2 t}$. To determine stability, we consider the limit $t \to \infty$ and if the solution approaches the fixed point $(0,0)$. With both eigenvalues negative, we can assume without loss of generality that $\lambda_2 < \lambda_1 < 0$. Hence, we see that each of the exponentials decays to zero as t grows, that is $\mathbf{x}(t) \to \mathbf{0}$; we know the fixed point is **stable**. Since $\lambda_2 < \lambda_1$, the leading order of the solution is the first term, that is $\mathbf{x}(t) \sim A\mathbf{v}_1 e^{\lambda_1 t}$. In other words, the solution approaches $(0,0)$ along the direction of the eigenvector with largest eigenvector; the fixed point is a **node** therefore because the limit is approached in a definite direction. We provide a sketch in Figure 5 below.

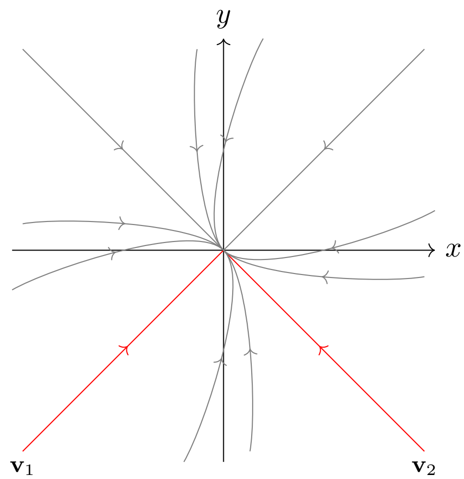

Figure 5: The phase portrait of a stable node.

(ii) Here, the eigenvalues are real so this is also Case (i) of the previous theorem; the general solution is again $\mathbf{x}(t) = A\mathbf{v}_1 e^{\lambda_1 t} + B\mathbf{v}_2 e^{\lambda_2 t}$. We again consider the limit $t \to \infty$. Since both eigenvalues are positive here, the exponentials diverge to infinity as t grows, meaning $\mathbf{x}(t) \to \infty$; we know the fixed point is **unstable**. Hence, everything is the same as in (i) except the orientations are reversed; it is again a **node**; see Figure 6 below.

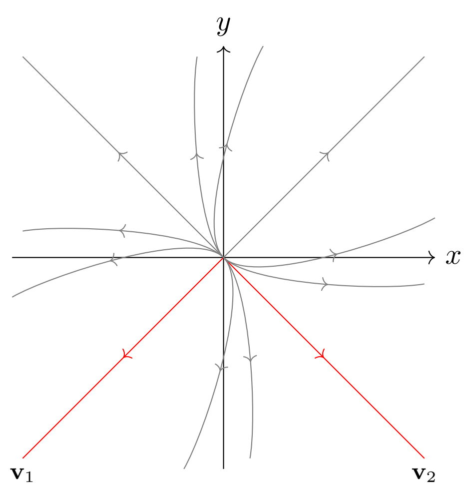

Figure 6: The phase portrait of an unstable node.

(iii) We are once again in Case (i) of the previous theorem, so the general solution has the same form as before: $\mathbf{x}(t) = A\mathbf{v}_1 e^{\lambda_1 t} + B\mathbf{v}_2 e^{\lambda_2 t}$. Note we assume that $\lambda_1 > 0$ and $\lambda_2 < 0$. In the limit $t \to \infty$, we have the leading order $\mathbf{x}(t) \sim A\mathbf{v}_1 e^{\lambda_1 t}$, the one with positive eigenvalue. As such, the fixed point is **unstable** since this blows up. However, in the limit $t \to -\infty$, we have the leading order $\mathbf{x}(t) \sim B\mathbf{v}_2 e^{\lambda_2 t}$, the one with negative eigenvalue. This blows up but from a different direction, so we have a **saddle**. We see this in Figure 7 below.

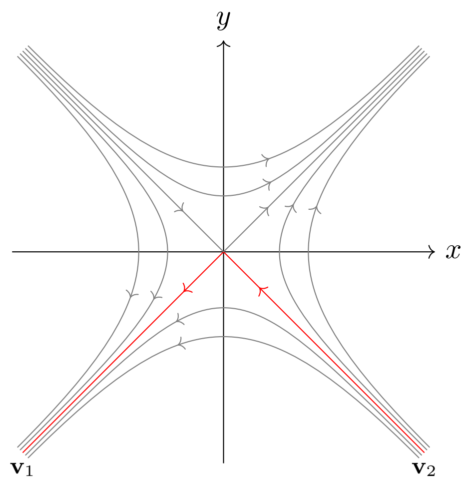

Figure 7: The phase portrait of an (unstable) saddle.

(iv) Since the eigenvalues are complex, this falls under Case (iii) of the previous theorem, meaning the general solution is $\mathbf{x}(t) = e^{pt}\big((A\mathbf{a} + B\mathbf{b})\cos(qt) + (B\mathbf{a} - A\mathbf{b})\sin(qt)\big)$. The positions $x(t)$ and $y(t)$ oscillate according to a sinusoidal function; the solution periodically travels through the four quadrants of the xy-plane, the time to complete one orbit is $2\pi/q$. Having real parts zero means $\|\mathbf{x}(t)\|^2 = A^2 + B^2$; we have a family of circles centred at $(0,0)$. The fixed point is a **centre** (which is **neutrally stable**) as pictured in Figure 8 below.

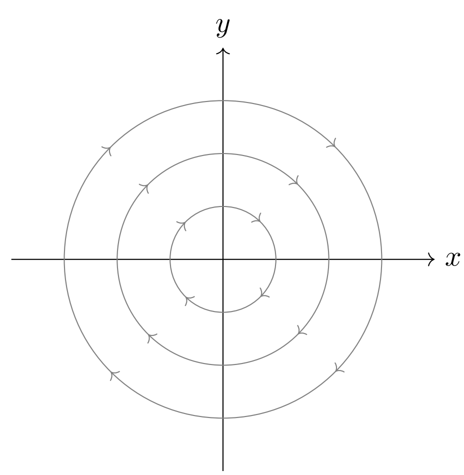

Figure 8: The phase portrait of a centre.

> **Note:** If the eigenvalues are $\pm ki$ with $k \in \mathbb{R}$, the trajectories are concentric **ellipses**.

(v) Here, the eigenvalues are complex so this is also Case (iii) of the previous theorem; the general solution is again $\mathbf{x}(t) = e^{pt}\left((A\mathbf{a} + B\mathbf{b})\cos(qt) + (B\mathbf{a} - A\mathbf{b})\sin(qt)\right)$. Since the real part $p < 0$, the exponential will decrease as $t \to \infty$, meaning the solution approaches $(0, 0)$. This implies the fixed point is **stable**. Note the trajectories 'spin' around the fixed point, so there is not a definite direction (this is a **spiral**). We see an example in Figure 9 below.

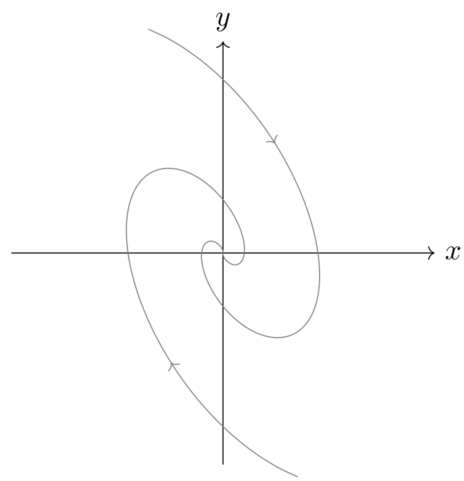

Figure 9: The phase portrait of a stable spiral.

(vi) Lastly, the eigenvalues are again complex so we are in Case (iii) of the previous theorem once more; the general solution is still $\mathbf{x}(t) = e^{pt}\left((A\mathbf{a} + B\mathbf{b})\cos(qt) + (B\mathbf{a} - A\mathbf{b})\sin(qt)\right)$. The reasoning in (v) still holds except the directions of the trajectories are reversed. Hence, we obtain an **unstable spiral**; the fact that the real part $p > 0$ means the exponential will increase as $t \to \infty$ and thus we move away from the fixed point; see Figure 10 below. $\square$

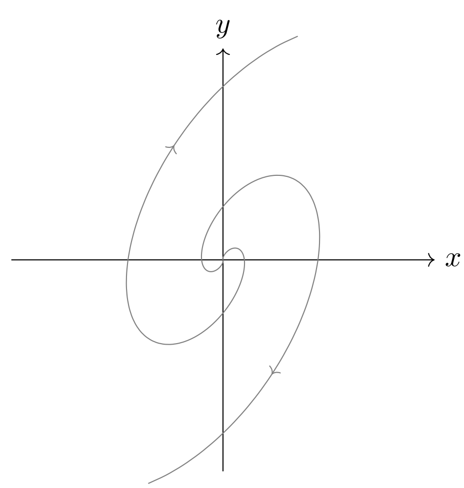

Figure 10: The phase portrait of an unstable spiral.

Method – Classifying a Critical Point: Suppose we are given a system of linear first-order ODEs of the form (5.2) and we wish to classify the type of fixed point that is $(0, 0)$.

(i) Write the system in matrix form (5.3).
(ii) Find the eigenvalues λ_1 and λ_2 of the matrix $\boldsymbol{A}$.
(iii) Use the classification theorem to determine the nature of the fixed point.

5.3 The Relationship Between Two-Variable ODEs and Second-Order ODEs

We shall now see that a single (one-dimensional) second-order ODE with constant coefficients can be converted into a system of two-dimensional first-order ODEs.

Proposition *Consider the following second-order ODE where a and b are constants:*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = 0. \tag{5.7}$$

This is equivalent to the following two-dimensional system of first-order ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \qquad \textit{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -bx - ay.$$

Proof: We shall here use the dot notation for the derivatives. Indeed, the given second-order ODE (5.7) can be written as $\ddot{x} + a\dot{x} + bx = 0$. We next define a new dependent variable by

$$\dot{x} =: y$$

Consequently, we see that $\ddot{x} = \dot{y}$; we can therefore write (5.7) as

$$\dot{y} = -bx - ay.$$

Thus, we obtain an ODE for $x(t)$ and an ODE for $y(t)$; this is precisely what we want. □

Note: In matrix notation, the above proposition tells us that $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Method – Phase Portraits for Second-Order ODEs: If we are given a second-order ODE (with constant coefficients), we can sketch its phase portrait by following this method.

(i) Use the above proposition to form a system of first-order ODEs.
(ii) Solve the system of first-order ODEs.
(iii) Classify the fixed point by using the previous theorem.
(iv) Use Steps (ii) and (iii) to piece together what the trajectories look like.

6 Application of Second-Order ODEs: Harmonic Oscillation

Next, we use the theory and solution methods of second-order ODEs from §4 in conjunction with phase plane methods from §5 to a case study: harmonic motion. As in §3, the goal is not only to solve an ODE but to formulate one by using appropriate modelling assumptions.

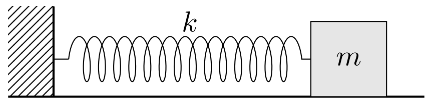

Figure 11: A horizontal spring-mass system.

Throughout, we consider the system in Figure 11 above, where a mass m resting on a horizontal surface is connected to a spring. If m is pulled to the right and then released, it will move horizontally under the action of the spring (in the leftward direction towards the wall).

> **Definition** The equilibrium position of the spring is where it is **not** extended/compressed.

We must now set up a coordinate system. Indeed, let x be the (horizontal) position of the mass m. We put the origin $x = 0$ when the spring is at its equilibrium position. By convention, set the rightward direction to be positive. The velocity of the mass at some time t is

$$v(t) = \dot{x}(t).$$

The acceleration, which is the rate of change of velocity, at some time t is therefore

$$a(t) = \dot{v}(t) = \ddot{x}(t).$$

> **Note:** We assume the mass m is constant. We therefore know how the net force F acting upon it changes as a function of position and time; note that $F = F(x, t)$. Using Newton's Second Law of Motion, we can formulate an ODE which we will eventually solve.

6.1 Simple Harmonic Motion

First, we assume there is **no** friction between the mass and the horizontal surface. In this case, there are three forces acting upon the mass: gravity; the force due to the spring; the normal reaction from the horizontal surface. Because the surface is horizontal, the normal reaction and gravity cancel each other out. In other words, there is no vertical motion and the mass moves horizontally only due to the spring. We now give this force a particular name.

> **Definition** The force due to the spring acting on the mass is called the tension, T.

> **Definition** Suppose we have a horizontal spring-mass system as in Figure 11. Then, Hooke's Law states that tension scales linearly with that distance, that is $T = -kx$ where $k > 0$ is a positive constant called the spring constant.

Note: We have a negative sign in Hooke's Law since we chose rightward direction to be positive, and tension always acts in the opposite direction of the deformation of the spring.

Definition The natural (angular) frequency of a spring as in Figure 11 is $\omega_0 := \sqrt{k/m}$.

Proposition *Consider a horizontal spring-mass system as in* Figure 11*. The displacement of the mass is given by* $x(t) = x_0 \cos(\omega_0 t)$*, where* ω_0 *is the natural frequency and* $x(0) = x_0$*.*

Proof: Using Newton's Second Law of Motion, we know the position $x(t)$ of the spring satisfies

$$T = ma \qquad \Leftrightarrow \qquad m\ddot{x} + kx = 0,$$

by using Hooke's Law for the tension. Since both k and m are positive, the natural frequency ω_0 is well-defined and we can use it to re-write the above ODE in the following way:

$$\ddot{x} + \omega_0^2 x = 0.$$

This is a homogeneous linear second-order ODE with constant coefficients; this can be solved using the method presented in §4.1.1. Indeed, the characteristic equation is

$$\lambda^2 + \omega_0^2 = 0 \qquad \Rightarrow \qquad \lambda = \pm i\omega_0.$$

Case (iii) of the theorem in §4.1.1 tells us the general solution of the ODE, i.e.

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

The fact that $x(0) = x_0$ implies that $A = x_0$ (called the amplitude). We need another condition to determine the other arbitrary constant; we know the mass is released from rest which is to say that $\dot{x}(0) = 0$. Differentiating the general solution tells us that

$$\dot{x}(t) = \omega_0 B\cos(\omega_0 t) - \omega_0 A\sin(\omega_0 t).$$

Thus, $\dot{x}(0) = 0$ implies that $B = 0$; we get the intended general solution $x(t) = x_0\cos(\omega_0 t)$. □

6.2 Damped Harmonic Oscillation

Next, we assume there **is** friction, f, between the mass and the horizontal surface. Note that friction always opposes the motion and, if the velocity is not too large, then it can be assumed to be proportional to velocity. Hence, we have this relation, where $b > 0$ is constant:

$$f = -bv = -b\dot{x}.$$

Definition The damping coefficient is $\mu := b/2m$, where b is the friction constant.

Proposition *Consider a horizontal spring-mass system as in* Figure 11 **with** *friction.*

(i) *If $\mu > \omega_0$, then the displacement is*

$$x(t) = e^{-\mu t}\left(Ae^{\sqrt{\mu^2-\omega_0^2}t} + Be^{-\sqrt{\mu^2-\omega_0^2}t}\right).$$

(ii) *If $\mu = \omega_0$, then the displacement is*

$$x(t) = e^{-\mu t}(A + Bt).$$

(iii) *If $\mu < \omega_0$, then the displacement is*

$$x(t) = e^{-\mu t}\left(A\cos\left(\sqrt{\omega_0^2-\mu^2}t\right) + B\sin\left(\sqrt{\omega_0^2-\mu^2}t\right)\right).$$

Proof: We again begin by forming a second-order ODE, again using Newton's Second Law of Motion but noting that the net force here is tension and friction, that is

$$T + f = ma \qquad \Leftrightarrow \qquad m\ddot{x} + b\dot{x} + kx = 0.$$

Using both the natural frequency and the damping coefficient, the ODE can be re-written as

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = 0.$$

This is again a homogeneous linear second-order ODE with constant coefficients; per the usual method, we look at the characteristic equation and find its roots:

$$\lambda^2 + 2\mu\lambda + \omega_0^2 = 0 \qquad \Rightarrow \qquad \lambda = \mu \pm \sqrt{\mu^2 - \omega_0^2}.$$

As with the theorem in §4.1.1, $x(t)$ will depend on the sign of the discriminant $\Delta \coloneqq \mu^2 - \omega_0^2$.

(i) If $\mu > \omega_0$, then we have two real roots $\lambda_1 = -\mu + \sqrt{\mu^2 - \omega_0^2}$ and $\lambda_2 = -\mu - \sqrt{\mu^2 - \omega_0^2}$. Hence, the general solution of the ODE is precisely Case (i) of the aforementioned theorem and this produces exactly what is written in the statement of the proposition.

(ii) If $\mu = \omega_0$, then we have one real repeated root $\lambda = -\mu$. Here, the general solution of the ODE is given by Case (ii) of the aforementioned theorem and we obtain exactly the solution we expect.

(iii) If $\mu < \omega_0$, then we have two complex roots $\lambda_1 = -\mu + i\sqrt{\omega_0^2 - \mu^2}$ and $\lambda_2 = -\mu - i\sqrt{\omega_0^2 - \mu^2}$. Again, the general solution of the ODE is given by Case (iii) of the aforementioned theorem and it gives us precisely what is written in the statement of the result. □

Note: We call Case (i) overdamped, Case (ii) critically damped and Case (iii) underdamped.

We see an idea of what solutions of damped harmonic motion look like in Figure 12 below.

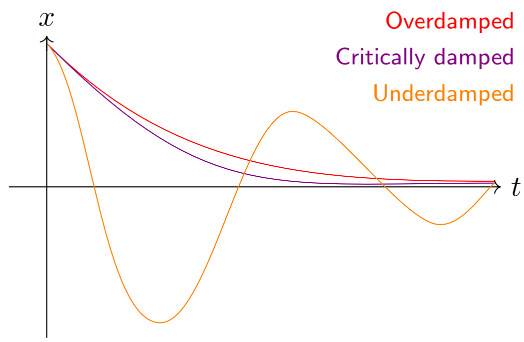

Figure 12: Solutions of damped harmonic motion.

6.3 Forced Oscillation and Resonance

The final thing we look at is when the mass is being driven by a time-dependent external force $F = F(t)$. For simplicity, we assume the dampening $\mu = 0$ and that the external force looks like

$$F(t) = F_0 \cos(\omega t),$$

for $F(0) = F_0$ an initial force and ω an imposed frequency (not necessarily the same as ω_0).

> **Proposition** *Consider a horizontal spring-mass system as in* Figure 11 *with force* $F(t)$*.*
>
> (i) *If* $\omega \neq \omega_0$*, then the displacement is*
>
> $$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2}\cos(\omega t).$$
>
> (ii) *If* $\omega = \omega_0$*, then the displacement is*
>
> $$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) - \frac{F_0}{2\omega_0} t \sin(\omega_0 t).$$

Proof: We again apply Newton's Second Law of Motion to obtain an ODE describing the motion:

$$T + F(t) = ma \qquad \Leftrightarrow \qquad \ddot{x} + \omega_0^2 x = F_0 \cos(\omega t).$$

This is an inhomogeneous linear second-order ODE with constant coefficients; we know from §4.2.1 that the general solution will consist of a complementary function and a particular integral. Recall that the complementary function is a general solution of the **homogeneous** equation, meaning it is precisely that which we obtained in the previous proposition in §6.2:

$$x_C(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

Using the method in §4.2.1 and noting $h(t) = F_0\cos(\omega t)$ contains sinusoids, our trial x_P is

$$x_P(t) = a\cos(\omega t) + b\sin(\omega t). \qquad \textbf{(6.4)}$$

(i) Let $\omega \neq \omega_0$. In this case, (6.4) is **not** in the complementary function; it is independent and therefore will work as is. To determine the coefficients a and b, we use the usual method of substituting it into the ODE and solving for them. We need only the second derivative:

$$\ddot{x}_P = -\omega^2 a\cos(\omega t) - \omega^2 b\sin(\omega t).$$

Substituting this into the left-hand side of the ODE gives us

$$(-\omega^2 a + \omega_0^2 a)\cos(\omega t) + (-\omega^2 b + \omega_0^2 b)\sin(\omega t) = F_0\cos(\omega t).$$

From this, we compare the coefficients and conclude that

$$\left.\begin{aligned}(\omega_0^2 - \omega^2)a &= F_0\\ (\omega_0^2 - \omega^2)b &= 0\end{aligned}\right\} \quad\Rightarrow\quad \begin{cases} a = \dfrac{F_0}{\omega_0^2 - \omega^2}\\ b = 0\end{cases}.$$

Thus, the particular solution has the form $x_P(t) = F_0/(\omega_0^2 - \omega^2)\cos(\omega t)$ and the general solution is exactly as expected. (i.e. as written in the statement of the proposition).

(ii) Let $\omega = \omega_0$. In this case, (6.4) **is** in the complementary function and so it does not work! However, we can follow the usual rule of multiplying it by t to obtain a new trial:

$$x_P(t) = \big(a\cos(\omega t) + b\sin(\omega t)\big)\, t.$$

We again substitute this into the ODE and compare coefficients. The second derivative is

$$\ddot{x}_P = \Big(-a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t)\Big)\, t + 2\left(-a\omega\sin(\omega t) - b\omega\cos(\omega t)\right).$$

Substituting this into the left-hand side of the ODE gives us

$$2\left(-a\omega\sin(\omega t) - b\omega\cos(\omega t)\right) = F_0\cos(\omega t).$$

We again compare the coefficients and conclude that

$$\left.\begin{aligned} a &= 0\\ -2b\omega &= F_0\end{aligned}\right\} \quad\Rightarrow\quad \begin{cases} a = 0\\ b = -\dfrac{F_0}{2\omega}\end{cases}.$$

Consequently, the particular solution is $x_P(t) = -F_0/(2\omega)t\sin(\omega t)$. Since $\omega = \omega_0$, we can use the latter and this gives us precisely the general solution we want. □

Note: If $\omega \approx \omega_0$, the amplitude blows up and gets infinitely larger; this phenomenon is known as resonance. In fact, this idea is captured in Case (ii) of the proposition. Thus, if the forcing frequency is at, or close to, the natural frequency, it generates a huge oscillation.

In other words, if you are pushing a frictionless swing, each time you push it synchronously with its natural oscillation, you add energy to the system and it will grow in amplitude. If instead you push it asynchronously with its natural oscillation, you will occasionally slow the swing down, causing the amplitude to remain bounded.