MATH1225 Introduction to Geometry

Cheatsheet

2023/24

This document collects together the important definitions and results presented throughout the lecture notes. The numbering used throughout will be consistent with that in the lecture notes.

Contents

1 Elementary Plane Geometry

1.2 Statements

Definition A statement is a collection of sentences that is either true or false.

Note: A common type of mathematical statement is of the form "if P, then Q", which is the same as "P implies Q", where P and Q are themselves statements. This is denoted $P \Rightarrow Q$. We commonly call statement P the assumption and statement Q the conclusion.

Remark The statement $P \Rightarrow Q$ is **always** true if the assumption P is false. This may be confusing at first glance but we will look at some ways to see this. Consider the statement "if I put money into a vending machine, then it gives me a snack" and look at the truth of the conclusion:

(i) If Q is true (I put **no** money in and **get** a snack), this doesn't violate $P \Rightarrow Q$.

(ii) If Q is false, (I put **no** money in and **don't** get a snack), this also doesn't violate $P \Rightarrow Q$.

1.3 Lines and Right Angles

Definition A line is an infinitely-long object with no curvature, depth or width, whereas a line segment is the straight path between two points.

Definition A right angle is defined as one of the angles formed when a line segment ends on a line, forming two equal angles; this is pictured in Figure 1. In this case, the line and line segment are called perpendicular (or orthogonal).

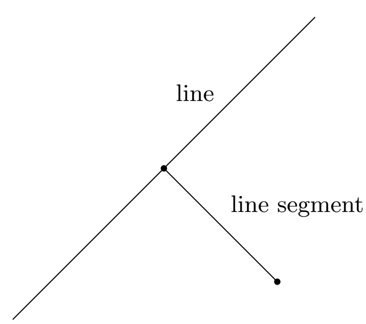


Figure 1: The geometric picture of a right angle.

Definition Two distinct lines are parallel if they do **not** intersect each other.

Note: We can say two line **segments** are parallel if and only if the lines they extend to are parallel. However, this relies on first making an assumption (we discuss this below).

1.4 Axioms

In general, an *axiom* is a rule that we impose before we start delving into mathematical theory.

Axiom (Euclidean Axioms) *The Euclidean axioms in modern-day language are as follows:*
(E1) *There is a unique line segment between any two distinct points.*
(E2) *Any line segment can be extended to a line.*
(E3) *There is a circle at point P with radius r, for any P and length r line segment at P.*
(E4) *All right angles are equal.*
(E5) *For a line L and a point P* **not** *on L, there is a unique line through P parallel to L.*

Remark Axiom (E5) is the *Parallel Postulate*. There was a time where mathematicians did not know if it was redundant: could (E5) be deduced from the others? The answer is **no**. In fact, if we remove (E5), we get more exotic forms of geometry (hyperbolic/elliptic geometry).

(a) Axiom (E2).　　(b) Axiom (E5).

Figure 2: The geometric interpretation of the Euclidean Axioms.

1.5 Line Segments and Rays

Note: Throughout, we label a line segment by its endpoints: if a segment connects A and B, it is denoted AB. We also do this for the line we get by extending the segment AB.

Definition Let L be a line segment containing a point P. Then, a ray from P is the subset which starts at P and extends infinitely along the rest of the line, as in Figure 3.

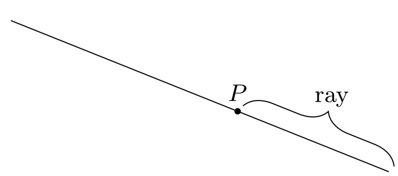


Figure 3: The geometric picture of a ray.

Definition Let A, B, C be points. They are collinear if there is a line on which they all lie.

Remark In light of this definition, we can combine Axioms (E1) and (E2) to conclude that any two points are collinear. This is of course not true in general for three points (if it was, triangles wouldn't exist)! But the case of three points pushes forward our thinking to looking at angles.

Note: We measure angles in **radians** (not degrees) in this module, unless stated otherwise.

Definition Let A, B, C be non-collinear points. The non-reflexive angle between the rays AB and AC is some number $\angle ABC \in (0, \pi)$.

This number $\angle ABC$ we call the non-reflexive angle satisfies the following properties:

- If B is between A and C and the three are collinear, we have $\angle ABC = \pi$.
- If B is **not** between A and C and the three are collinear, we have $\angle ABC = 0$.

Definition Let A, B, C be non-collinear points. The triangle ABC is the shape formed with line segments AB, BC, CA. The points A, B, C are then called the vertices of the triangle and the aforementioned line segments are the edges of the triangle.

Remark Being really technical, we should say that A, B, C are the *triangle* and the line segments AB, BC, CA are the *trilateral*. However, there is no real need to distinguish the two in this module and thus we use "triangle" to mean any picture of the form in Figure 4.

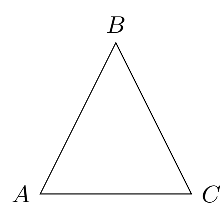


Figure 4: A set of three non-collinear points forming a triangle.

1.6 Congruent Triangles

Definition 1.6.1 Triangles ABC and DEF are congruent if there is some correspondence between vertices such that corresponding edges are equal in length and corresponding angles are equal in size. We denote this by $\triangle ABC \equiv \triangle DEF$.

Reminder: An equivalence relation $\sim$ on a set X is a relation that satisfies the following:

(i) For all $x \in X$, we have $x \sim x$. (**Reflexivity**)
(ii) For all $x, y \in X$, we have $x \sim y$ implies $y \sim x$. (**Symmetry**)
(iii) For all $x, y, z \in X$, we have $x \sim y$ and $y \sim z$ imply $x \sim z$. (**Transitivity**)

Proposition *Congruence of triangles is an equivalence relation on the set of triangles.*

Sketch of Proof: Show the equivalence relation axioms directly by unpicking Definition 1.6.1. ☐

Axiom (Side-Side-Side) *Let ABC and DEF be triangles that satisfy $AB = DE$ and $BC = EF$ and $CA = FD$. Then, we have $\triangle ABC \equiv \triangle DEF$. In other words, triangles with three common side lengths are congruent.*

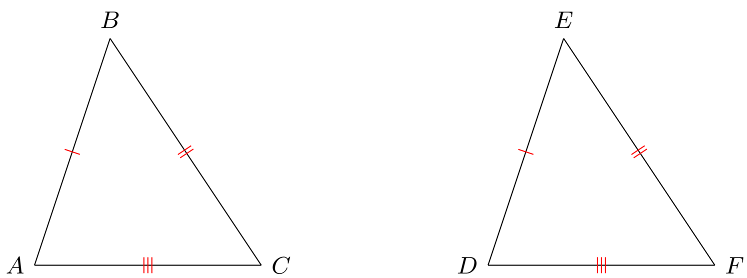


Figure 5: The geometric picture interpretation of the Side-Side-Side (SSS) Axiom.

Axiom (Side-Angle-Side) *Let ABC and DEF be triangles that satisfy $AB = DE$ and $BC = EF$ and $\angle ABC = \angle DEF$. Then, we have $\triangle ABC \equiv \triangle DEF$. In other words, triangles with two common side lengths and a common angle between them are congruent.*

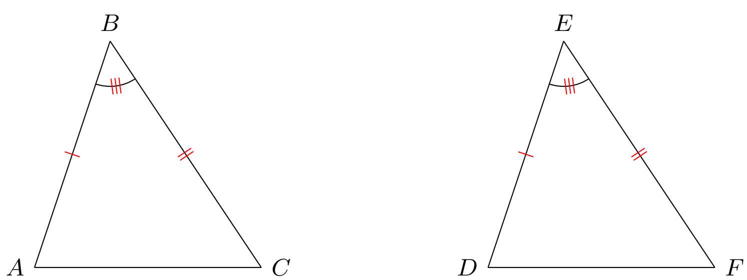


Figure 6: The geometric picture interpretation of the Side-Angle-Side (SAS) Axiom.

Theorem (Angle-Angle-Side) *Let ABC and DEF be triangles with $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$ and $CA = FD$. Then, we have $\triangle ABC \equiv \triangle DEF$. In other words, triangles with two common angles and a side length* **not** *between them are congruent.*

Proof: Omitted; this is similar to a result in one of the exercise sheets. ☐

1.7 Complementary, Supplementary and Vertically Opposite Angles

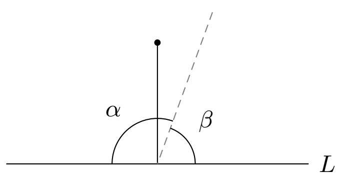
Definition 1.7.1 Suppose we have two angles α and β.

(i) They are complementary if $\alpha + \beta = \pi/2$.

(ii) They are supplementary if $\alpha + \beta = \pi$.

Lemma 1.7.2 *Angles on a line are supplementary*

Proof: Consider a line L with angles $\alpha \geq \beta$ (without loss of generality) as drawn below.



By Axiom (E4), all right angles are equal. Now, we can draw a line segment that meets L at right angles. This shows us that we can decompose $\alpha + \beta$ into $\alpha_1 + \alpha_2 + \beta$ where $\alpha_2 + \beta = \pi/2$ are complementary and α_1 is a right angle (and thus equals $\pi/2$ also). □

Remark 1.7.3 It may seem like the statement of Lemma 1.7.2 is obvious. However, the point of this is to prove that even something seemingly trivial that is itself **not** an assumption does indeed follow from the axioms we set out at the beginning.

Definition 1.7.4 Suppose we have two lines crossing at a point P. We call the non-adjacent angles (the ones looking across from one another as in Figure 7) vertically opposite.

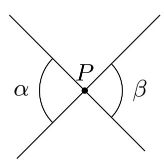
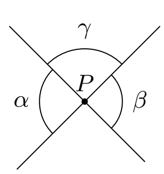


Figure 7: A pair of vertically opposite angles.

Proposition 1.7.5 *Vertically opposite angles are equal.*

Proof: Consider again Figure 7 but we embellish the picture by labelling the upper angle:



By Lemma 1.7.2, we can see that $\alpha + \gamma = \pi$ and that $\gamma + \beta = \pi$. Subtracting the second equation from the first yields $\alpha - \beta = 0$, from which we conclude the indented result: $\alpha = \beta$. □

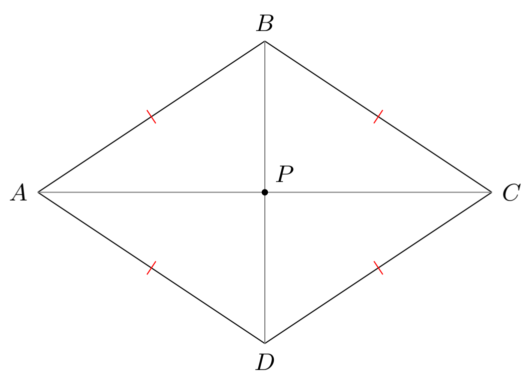
Note: This is an example of a *direct proof*; this is where we start with the assumptions, which imply further statements that ultimately imply the conclusion. There is no trickery here: what you see is what you get.

1.8 Rhombus

Definition A rhombus $ABCD$ is the shape formed from four points A, B, C, D such that any three of them are non-collinear with sides $AB = BC = CD = DA$.

Proposition *The diagonals of a rhombus* bisect *each other, meaning split exactly in half, and meet at right angles.*

Proof: We draw an example rhombus and label the intersection point of the diagonals by P.



We now look at some of the constituent triangles formed by drawing in the diagonals:

- We see the congruence $\triangle ABD \equiv \triangle BCD$ as a result of SSS.
- We see the congruence $\triangle APD \equiv \triangle CPD$ as a result of SAS.

Now then, $AP = CP$ since these sides are in correspondence when viewing the two congruent triangles they belong to. In other words, the diagonal BD has bisected the diagonal AC. An identical argument tells us that $BP = DP$. Moreover, $\angle APD = \angle CPD$ because these are the angles in correspondence when looking at the congruent triangles they belong to. However, Lemma 1.7.2 tells us that they are supplementary: $\angle APD + \angle CPD = \pi$. Because they are equal, they are both right angles. An identical argument applies to give us $\angle BPA = \angle BPC$ and these are also supplementary. □

1.9 Transversals and Alternate Angles

Definition 1.9.1 A line K is a transversal of the distinct lines L_1 and L_2 if it crosses them.

Note: It will be helpful to keep a picture in mind moving forward, and set-up some notation for the various angles involved. We therefore draw the picture in Figure 8 below.

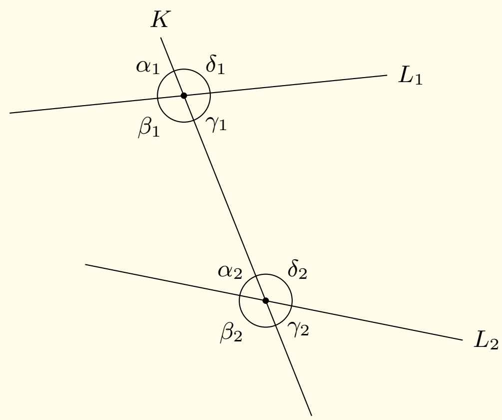
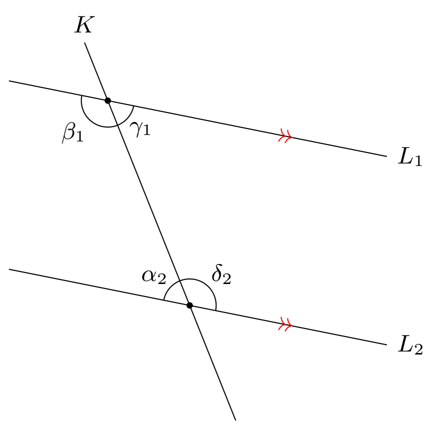


Figure 8: A transversal K of lines L_1 and L_2.

Theorem 1.9.2 (Alternate Angle Theorem) *If L_1 and L_2 are distinct parallel lines with transversal K, then each pair of* alternate angles *is equal, meaning $\alpha_2 = \gamma_1$ and $\beta_1 = \delta_2$.*

Proof: Because we are assuming that L_1 and L_2 are parallel, we can draw a slightly different picture to that in Figure 8. For ease, we only consider the four angles of interest to us here.



It suffices to prove only one of them, say $\beta_1 = \delta_2$, because the same argument can be adapted to show the other one. Indeed, assume to the contrary that $\beta_1 \gneqq \delta_2$ (where I use the "greater than and not equal to" symbol $\gneq$ just this once instead of the equivalent $>$ for emphasis). But now,

$$\beta_1 + \gamma_1 > \delta_2 + \gamma_1.$$

However, $\beta_1 + \gamma_1 = \pi$ by Lemma 1.7.2. Hence, we have $\delta_2 + \gamma_1 < \pi$ and Axiom (E5) guarantees that L_1 and L_2 must therefore meet. But this contradicts the parallel assumption we made. □

Note: This is an example of a *proof by contradiction*; this is where we assume the opposite of the conclusion we are aiming for and show that this results in some nonsense.

Remark 1.9.3 In the above proof, we used the following interpretation of Axiom (E5): if a line K meets two straight lines at two points P_1 and P_2 in such a way that the sum of the angles between K and each ray stemming from P_1 and P_2 is **less** than π, each ray/line will meet.

Theorem 1.9.4 (Corresponding Angle Theorem) *If L_1 and L_2 are distinct parallel lines with transversal K, then each pair of* corresponding angles *are equal, meaning $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$ and $\delta_1 = \delta_2$.*

Proof: Consider again the picture in Figure 8. Then, we know the Alternate Angle Theorem (Theorem 1.9.2) guarantees $\alpha_2 = \gamma_1$ and $\beta_1 = \delta_2$. We can also use Proposition 1.7.5 to conclude that every pair of vertically opposite angles is equal: $\alpha_1 = \gamma_1$, $\beta_1 = \delta_1$, $\alpha_2 = \gamma_2$ and $\beta_2 = \delta_2$. Therefore, making the relevant substitutions produces the four equations we want. □

1.10 Converses

Definition 1.10.1 Consider a statement of the form "if P, then Q", where P and Q are themselves statements. The converse of this statement is "if Q, then P".

Note: Using the notation introduced at the beginning, the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. The fact one of these statements may be true has **no** impact on the truth of the other.

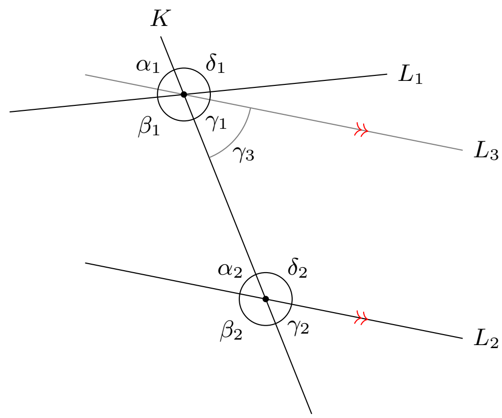
It turns out the converse to the Alternate Angle Theorem (Theorem 1.9.2) is actually true!

Proposition 1.10.3 *Let L_1 and L_2 be distinct lines with a transversal K. If each pair of alternate angles is equal, then L_1 and L_2 are parallel.*

Proof: Omitted; we prove a stronger result below. □

Proposition 1.10.4 *Let L_1 and L_2 be distinct lines with a transversal K. If* **one** *pair of alternate angles is equal, then L_1 and L_2 are parallel.*

Proof: Consider the same picture as in Figure 8 and, without loss of generality, let $\alpha_2 = \gamma_1$ be the pair of equal alternate angles we have by assumption (note that the same works if we let $\beta_1 = \delta_2$ just by swapping notation). Denote by P the intersection between L_1 and K. By Axiom (E5), there exists a unique line L_3 through P that is parallel to L_2. We denote the angle between K and this new line L_3 by γ_3; this is all pictured below.

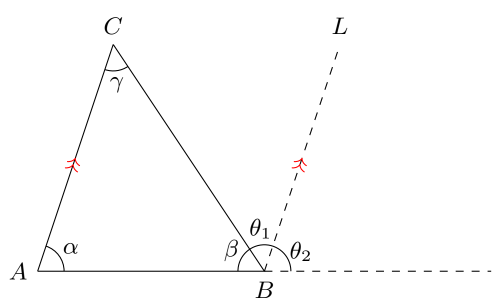


Because L_1 and L_3 are parallel, we can apply the Corresponding Angle Theorem (Theorem 1.9.4) to conclude that the alternate angles are equal. This means $\alpha_2 = \gamma_3$. Since we assumed that $\alpha_2 = \gamma_1$, we must have $\gamma_1 = \gamma_3$; this tells us that $L_3 = L_1$ and in fact the lines were parallel to begin with. □

> **Note:** We said that Proposition 1.10.4 is "stronger" than Proposition 1.10.3 just before it. This name is down to the fact that stronger results imply weaker results, and clearly **if** Proposition 1.10.4 is true (it is, we just proved it), then Proposition 1.10.3 is also true.

> **Proposition 1.10.5** *The sum of the angles in a triangle is π.*

Proof: Let ABC be a triangle with angles α, β, γ at A, B, C respectively. By Axiom (E5), there is a unique line through B which is parallel to AC. The picture of this set-up is drawn below.



By the Alternate Angle Theorem (Theorem 1.9.2), we have $\theta_1 = \gamma$. But by the Corresponding

Angle Theorem (Theorem 1.9.4), we also have $\theta_2 = \alpha$. Finally, applying Lemma 1.7.2 produces

$$\pi = \beta + \theta_1 + \theta_2 = \beta + \gamma + \alpha. \qquad \square$$

1.11 Bisectors and Perpendiculars

Definition 1.11.1 Let AB be a line segment. A bisector is a line through the midpoint of AB, that is the line separating the segment into two segments of equal length. The perpendicular bisector is the bisector that passes through AB at right angles.

Method – Finding a Perpendicular Bisector: Let AB be a line segment.

(i) Choose a number $r > 0$ such that it is greater than half the length of AB.
(ii) Construct two circles of radius r with centres A and B.
(iii) The perpendicular bisector is the line through the two intersection points.

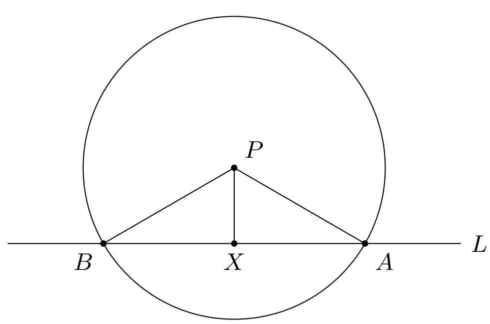
A similar notion to that above is that of an *angle bisector*, which is a line that divides an angle $\angle ABC$ formed by three distinct non-collinear points into two angles of equal size.

Method – Finding an Angle Bisector: Let $\angle ABC$ be an angle.

(i) Construct a circle of some fixed radius $r > 0$ centred at B.
(ii) Label the intersection of this circle with AB and BC by X and Y, respectively.
(iii) Construct two circles of the same radius r with centres X and Y.
(iv) The angle bisector is the line through B and the intersection points of X and Y.

Theorem 1.11.2 *Given a line L and a point P not on L, we can* drop a perpendicular *from P to L, meaning there is a line segment PX, with X on L, that is perpendicular to L.*

Proof: Draw a circle centred at P with radius large enough so that the circle intersects L at two distinct points, A and B say. We then perpendicularly bisect the line segment AB to determine the midpoint of this segment, namely X. This is pictured below.

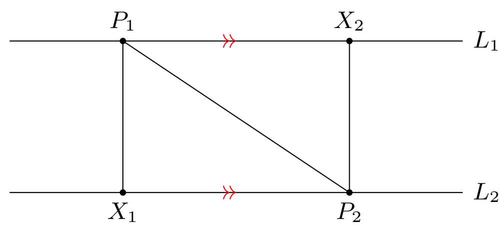


It remains to conclude that the perpendicular bisector also passes through P; this is done by drawing line segments PA and PB. Because PAB is an isosceles triangle, we can use SSS to conclude that $\triangle PXA \equiv \triangle PXB$. Because Lemma 1.7.2 guarantees that $\angle PXA + \angle PXB = \pi$, and the triangles are similar, it must be that $\angle PXA = \angle PXB = \pi/2$. □

Note: We define the distance $\mathrm{dist}(P, L)$ from a point P to a line L as the length of PX.

Proposition 1.11.3 *Let L_1 and L_2 be parallel lines. Then, the distance from any point on L_1 to L_2 is the same (and coincides with the distance from any point on L_2 to L_1). This is called the* distance *between the lines L_1 and L_2.*

Proof: Draw the parallel lines L_1 and L_2, which we annotate with more labels explained below.



Choose two points P_1 and P_2 on each of L_1 and L_2, respectively. By Theorem 1.11.2, we can drop a perpendicular from P_1 to L_2 (and similarly from P_2 to L_1). We label the foot of each perpendicular by X_1 and X_2, respectively. But we can now use the Angle-Angle-Side (AAS) theorem to conclude that $\triangle P_1P_2X_1 \equiv \triangle P_1P_2X_2$. Indeed, $\angle P_1X_1P_2 = \angle P_1X_2P_2 = \pi/2$ since they are dropped *perpendiculars*, and we know that $\angle X_1P_1P_2 = \angle X_2P_2P_1$ by the Alternate Angle Theorem, and of course they share a side P_1P_2. Therefore, we conclude $P_1X_1 = P_2X_2$. As we chose P_1 and P_2 arbitrarily, this argument works for **any** such points on each line. □

1.12 Pythagoras' Theorem and its Converse

Theorem 1.12.1 (Pythagoras' Theorem) *Let ABC be a right-angled triangle. Then, the square of the hypotenuse is equal to the sum of the squares of the other two sides.*

Proof: Let ABC be the right-angled triangle we work with as drawn and labelled below.

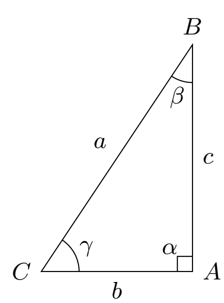
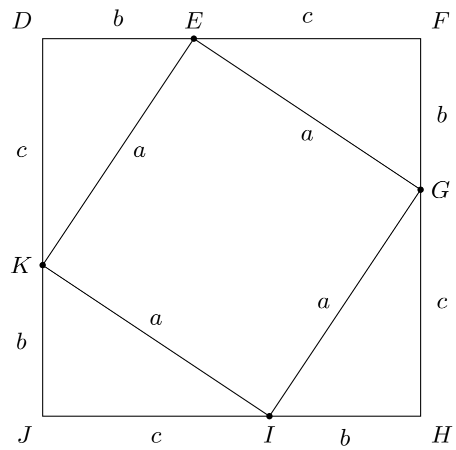


Figure 9: The right-angled triangle ABC.

We can now consider a square whose sides all have lengths $b+c$. If we mark some points on each side that separates each of them into two segments of lengths b and c, we can start to connect these new points as in the picture below.



We have $\triangle KDE \equiv \triangle EFG \equiv \triangle GHI \equiv \triangle IJK \equiv \triangle ABC$ by SAS, allowing us to label the internal connecting lines all by a. But now, the area of the square can be computed in two ways: the easy way is simply as $(b+c)^2$, since this is its side length. The second way is to sum the areas of the four triangles and the internal square. But these two things must be equal, so

$$(b+c)^2 = 4\left(\frac{1}{2}bc\right) + a^2.$$

Expanding the left and simplifying the right tells us that $a^2 + 2bc = b^2 + c^2 + 2bc$, from which we conclude that $a^2 = b^2 + c^2$. This is precisely the statement we wanted to show. □

Note: An immediate corollary of Pythagoras' Theorem is that the hypotenuse is longer than either of the other two sides. Indeed, since $a^2 = b^2 + c^2$, it is clear that $a^2 > b^2$ and $a^2 > c^2$. The final step is to take the square root, from which we conclude what we stated.

Corollary 1.12.2 *Let ABC be a right-angled triangle as in* Figure 9*. Then, $b+c > a$.*

Proof: Suppose to the contrary that $b+c \leq a$. Because these letters are side lengths, they are all positive. In particular, we have $b+c > 0$. Thus, squaring both sides tells us that

$$(b+c)^2 \leq a^2 \qquad \Rightarrow \qquad b^2 + c^2 + 2bc \leq a^2.$$

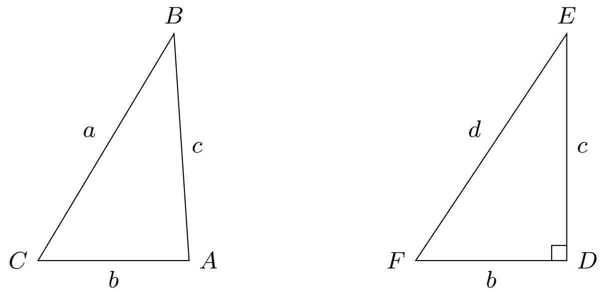
We now invoke Pythagoras' Theorem, which says $a^2 = b^2 + c^2$. Substituting this above yields

$$b^2 + c^2 + 2bc \leq b^2 + c^2 \qquad \Rightarrow \qquad 2bc \leq 0.$$

But this is a contradiction: recall that $b > 0$ and $c > 0$, so we must have $2bc > 0$. □

Theorem 1.12.3 (Converse of Pythagoras' Theorem) *Let ABC be a triangle with side lengths a, b, c opposite vertices A, B, C respectively where $a^2 = b^2 + c^2$. Then, ABC is a right-angled triangle with right angle $\angle CAB$.*

Proof: Let ABC be the following triangle, and consider also the right-angled triangle DEF which has two side lengths the same as the sides in ABC. This is pictured and labelled as follows.



We can apply Pythagoras' Theorem to triangle DEF in order to conclude that

$$d^2 = b^2 + c^2.$$

But we are assuming that $a^2 = b^2 + c^2$, so combining these equations tells us $a^2 = d^2$. Since we are dealing with side lengths, everything is positive and thus we have $a = d$. By SSS, we obtain $\triangle ABC \equiv \triangle DEF$, so ABC is in fact a right-angled triangle with $\angle CAB = \angle FDE = \pi/2$. $\square$

Method – Determining Right-Angledness: Suppose we have the side lengths a, b and c of a triangle, and assume without loss of generality that a is largest (if not, swap the labels).
(i) If $a^2 = b^2 + c^2$, then it **is** right-angled by the *converse* of Pythagoras' Theorem.
(ii) If $a^2 \neq b^2 + c^2$, then it's **not** right-angled by *Pythagoras' Theorem*.

1.13 Similar Triangles

Definition 1.13.1 Two triangles ABC and DEF are similar if the angles in ABC are the same as the angles in DEF. We denote this by $\triangle ABC \sim \triangle DEF$.

Note: Think of this as Angle-Angle-Angle (AAA); note it is **not** a congruence criteria.

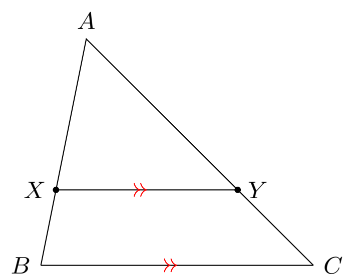
Remark The notion of similar triangles is weaker than that of congruence. As such, two congruent triangles are automatically similar to each other. However, these are different concepts: we can find a pair of similar triangles that are **not** congruent.

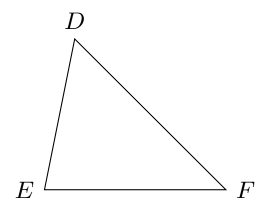
Reminder: The area of any triangle is given by $\frac{1}{2} \times$ base length $\times$ perpendicular height.

Proposition 1.13.2 *Let ABC and DEF be similar triangles. Then,*

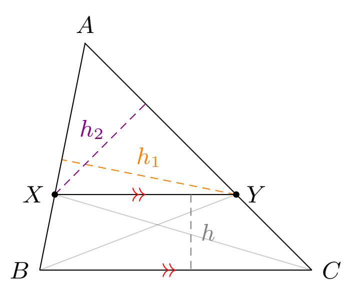
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

Proof: Without loss of generality, let $AB \geq DE$ (if not, just swap the labels on the triangles to make this argument work). The idea is to draw ABC and construct a triangle on it which is congruent to DEF. Choose a point X on the line AB such that $AX = DE$; we proceed similarly for a point Y on the line AC such that $AY = DF$. This is possible because $AB \geq DE$, so we can fit the length DE onto AB. This is pictured below, along with the triangle DEF for reference.





We know $\angle CAB = \angle FDE$ because the triangles are similar, so one can conclude from SAS that $\triangle DEF \equiv \triangle AXY$. In particular, we have $\angle AXY = \angle DEF$ since these two triangles are congruent. But again from our similarity assumption, we know $\angle DEF = \angle ABC$. So, we have a pair of alternate angles that are equal, so Proposition 1.10.3 tells us XY and BC are parallel. We now look at areas of some "sub-triangles" in this picture; we re-draw it and add more labels.



Drop perpendiculars h_1 from Y and h_2 from X. By the formula for area of a triangle, we have

$$\frac{AB}{DE} = \frac{AB}{AX} = \frac{\frac{1}{2}ABh_1}{\frac{1}{2}AXh_1} = \frac{\text{area}(ABY)}{\text{area}(AXY)} \qquad \text{and} \qquad \frac{AC}{DF} = \frac{AC}{AY} = \frac{\frac{1}{2}ACh_2}{\frac{1}{2}AYh_2} = \frac{\text{area}(ACX)}{\text{area}(AYX)}.$$

To conclude these are equal, it suffices to prove that $\text{area}(ABY) = \text{area}(ACX)$. Well, note that

$$\text{area}(ABY) = \text{area}(AXY) + \text{area}(XBY) \qquad \text{and} \qquad \text{area}(ACX) = \text{area}(AYX) + \text{area}(YCX),$$

so it is now sufficient to prove $\text{area}(XBY) = \text{area}(YCX)$; this is true as XBY and YCX have the same base XY and height h. The same argument works for showing $AB/DE = BC/EF$. $\square$

We will improve Proposition 1.13.2 in a moment, but first we will define another special type of statement which comes up a lot in mathematics. Fortunately, there isn't much to say because we have already been introduced to the concept of a converse earlier.

Definition Let P and Q be statements. The statement P if and only if Q is defined as the statement meaning **both** $P \Rightarrow Q$ and its converse $Q \Rightarrow P$. We denote this by $P \Leftrightarrow Q$.

It turns out the converse to Proposition 1.13.2 is also true; we state them both together below.

Theorem 1.13.3 *Let ABC and DEF be triangles. Then, they are similar if and only if*

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

Proof: ($\Rightarrow$) This is Proposition 1.13.2.

($\Leftarrow$) Suppose we have two triangles ABC and DEF where

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}. \tag{$\dagger$}$$

We will construct a new triangle XYZ such that the angles are equal to those in ABC (meaning they are similar: $\triangle XYZ \sim \triangle ABC$) and such that the side $XY = DE$. Since these triangles are similar by construction, we can use Proposition 1.13.2 to conclude that

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}. \tag{$\ddagger$}$$

Because $XY = DE$, we can combine ($\dagger$) and ($\ddagger$) at the first part in each equation to produce

$$\frac{BC}{EF} = \frac{BC}{YZ} = \frac{AC}{DF} = \frac{AC}{XZ}.$$

Therefore, we conclude that $EF = YZ$ and $DF = XZ$. As such, SSS implies $\triangle DEF \equiv \triangle XYZ$. We already established that $\triangle XYZ \sim \triangle ABC$, so we see that DEF is similar to ABC. $\square$

Note: In other words, Theorem 1.13.3 completely classifies similar triangles. Namely, two triangles are similar precisely when we can scale one of them up to get the other; the scale factor is the ratio in the statement of the theorem. Moreover, this is the **only** way to generate a triangle similar to a given one: scale it up or down (just not be zero or one).

1.14 Definitions of the Trigonometric Functions

Definition 1.14.2 Let ABC be a right-angled triangle as in Figure 9 and call $\theta := \angle BCA$. The sine of the angle θ is the real number

$$\sin(\theta) := \frac{AB}{BC} = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Proposition 1.14.1 *Let ABC and DEF be right-angled triangles whose right angles are $\angle CAB = \angle FDE = \pi/2$, and where we also assume $\angle BCA = \angle EFD$. Then,*

$$\frac{AB}{BC} = \frac{DE}{EF}.$$

Proof: As the sum of the angles in a triangle is π (Proposition 1.10.5), we have $\angle ABC = \angle DEF$. This means we have the similarity $\triangle ABC \sim \triangle DEF$. Finally, Theorem 1.13.3 applies to give us

$$\frac{AB}{DE} = \frac{BC}{EF} \qquad \Leftrightarrow \qquad \frac{AB}{BC} = \frac{DE}{EF}. \qquad \square$$

Note: Proposition 1.14.1 proves that $\sin(\theta)$ is *well-defined*, which means that it doesn't matter which triangle we choose in the definition. In general, something is well-defined if said thing seems like it depends on making a choice but, in reality, it does not.

We do the same for cosine and tangent, which we now introduce and prove are well-defined.

Definition Let ABC be a right-angled triangle as in Figure 9 and call $\theta := \angle BCA$. The cosine and the tangent of the angle θ are the respective real numbers

$$\cos(\theta) := \frac{AC}{BC} = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{and} \qquad \tan(\theta) := \frac{AB}{AC} = \frac{\text{opposite}}{\text{adjacent}}.$$

Proposition *Let ABC and DEF be right-angled triangles whose right angles are $\angle CAB = \angle FDE = \pi/2$, and where we also assume $\angle BCA = \angle EFD$. Then,*

$$\frac{AC}{BC} = \frac{DF}{EF} \qquad \textit{and} \qquad \frac{AB}{AC} = \frac{DE}{DF}.$$

Proof: As the sum of the angles in a triangle is π (Proposition 1.10.5), we have $\angle ABC = \angle DEF$. This means we have the similarity $\triangle ABC \sim \triangle DEF$. Finally, Theorem 1.13.3 applies to give us

$$\frac{BC}{EF} = \frac{AC}{DF} \quad \Leftrightarrow \quad \frac{AC}{BC} = \frac{DF}{EF} \qquad \text{and} \qquad \frac{AB}{DE} = \frac{AC}{DF} \quad \Leftrightarrow \quad \frac{AB}{AC} = \frac{DE}{DF}. \qquad \square$$

Remark The above definitions work when θ is an acute angle, so what about when it is a right-angle or an obtuse angle? Well, if $\theta = \pi/2$, we naturally define $\sin\big(\pi/2\big) = 1$ and $\cos\big(\pi/2\big) = 0$; think about what happens in the above definitions when $\theta \to \pi/2$ (and notice that $\tan\big(\pi/2\big)$ is undefined). Secondly, if θ is obtuse, this means it satisfies $\pi/2 < \theta < \pi$. Then, we can define a new angle $\Theta := \pi - \theta$, the so-called *supplement* of θ. Notice that $0 < \Theta < \pi/2$ is an acute angle, so we already have definitions for $\sin(\Theta)$, $\cos(\Theta)$ and $\tan(\Theta)$. Hence, we use these definitions:

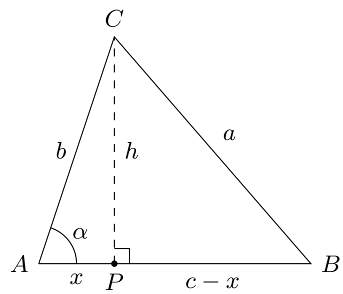
$$\sin(\theta) := \sin(\Theta), \qquad \cos(\theta) := -\cos(\Theta), \qquad \tan(\theta) := -\tan(\Theta).$$

1.15 Cosine Rule

Theorem 1.15.1 (Cosine Rule) *Let ABC be a triangle wit side lengths a, b, c opposite vertices A, B, C respectively with $\alpha \coloneqq \angle CAB$ the angle at vertex A. Then,*

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$

Proof: (i) Assume first that all angles in ABC are acute; this allows us to draw a nice picture of the triangle. From here, we drop a perpendicular from C to AB and label its foot by P. For notation, let $h \coloneqq CP$ and label $x \coloneqq AP$. This means that $BP = c - x$. This is sketched below.



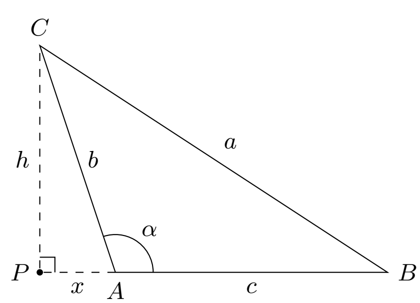
We now have two right-angled triangles ACP and BCP, to which we apply Pythagoras' Theorem:

$$b^2 = h^2 + x^2 \quad \text{and} \quad a^2 = h^2 + (c-x)^2 \qquad \Rightarrow \qquad a^2 - b^2 = c^2 - 2cx.$$

But using the definition of $\cos(\alpha) = x/b$, we get an expression in terms of the cosine for x which we can substitute above. Indeed, $x = b\cos(\alpha)$ and the above becomes

$$a^2 = b^2 + c^2 - 2cbc\cos(\alpha).$$

(ii) Assume next that α is obtuse; this allows us to draw a slightly different picture of the triangle. We then 'complete it' to a right-angled triangle by extending the base and dropping a perpendicular from C to the extended AB. This is drawn here.



We again apply Pythagoras' Theorem, this time to both ACP and BCP:

$$b^2 = h^2 + x^2 \quad \text{and} \quad a^2 = h^2 + (c+x)^2 \qquad \Rightarrow \qquad a^2 - b^2 = c^2 + 2cx.$$

But now, the definition of cosine applies to triangle ACP to give us the expression $x = b\cos(\pi - \alpha)$. Using what we talked about in the previous remark, we see that $\cos(\pi - \alpha) = -\cos(\alpha)$. Substituting this into the above equation produces

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$

(iii) Assume finally that $\alpha = \pi/2$. Then, $\cos(\alpha) = 0$ so this reduces to Pythagoras' Theorem. $\square$

2 Coordinate Geometry

2.1 Introduction

Reminder: The Cartesian product of sets A and B is $A \times B := \{(a, b) : a \in A \text{ and } b \in B\}$, that is the set consisting of pairs where the first entry lives in A and the second entry lives in B. If we take the Cartesian product of a set with itself a number of times, we write

$$A^n := A \times A \times \cdots \times A.$$

Instead of using axioms to discuss geometry, we can choose a *coordinate system* to specify points in the plane. One of the most common systems we see is the Cartesian coordinate system, named after René Déscartes who played an important role in the development of this theory.

Definition A coordinate system consists of making the following choices:

- Choose an origin, O.
- Choose perpendicular axes through O.
- Choose scales on each axis.
- Choose an orientation, that is an ordering on the axes.

For us, we work with a plane which can be described in coordinates by the Cartesian product

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}.$$

Here, the origin $O = (0, 0)$ and we have two axes: the x-axis consisting of the points $(x, 0)$ and the y-axis consisting of the points $(0, y)$. We can associate to any point in the plane a coordinate description (x, y) simply by dropping perpendiculars to each of the axes. On the other hand, we can construct a point (x, y) from any two real numbers by intersecting the perpendicular to the x-axis through $(x, 0)$ and the perpendicular to the y-axis through $(0, y)$.

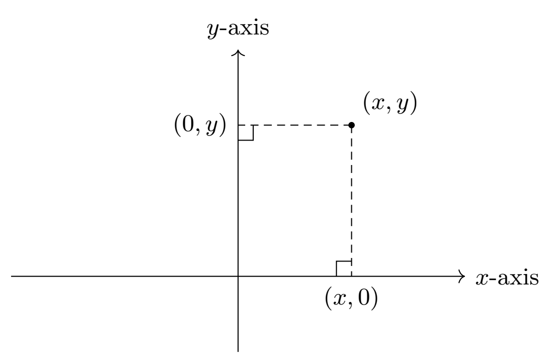


Figure 10: Cartesian coordinates for the plane $\mathbb{R}^2$.

2.2 Lines

Definition A (Cartesian) line in $\mathbb{R}^2$ is given by the equation $ax + by + c = 0$ for $a, b, c \in \mathbb{R}$.

Note: You have probably seen the equation $y = mx + c$ before as having defined a line. This isn't bad, but there is no way to write vertical lines $x = k$ for $k \in \mathbb{R}$ in this form. The point of the definition above is that we can write all such lines; we show this below:

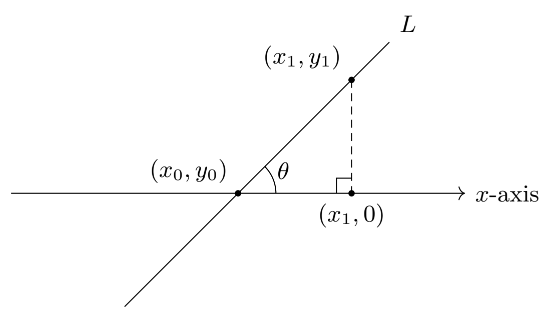
- To describe $x = k$, take $a = 1$, $b = 0$ and $c = k$ in the definition.
- To describe $y = mx + c$, take $a = m$, $b = -1$ and $c = c$ in the definition.

Remark Notice we say that $ax + by + c = 0$ can be used to define any line. But is the converse true, that is does every equation define a line? The answer turns out to be **no**. Indeed, if $a = 0$ and $b = 0$, we either get the empty set $\emptyset$ if $c \neq 0$ **or** we get all of $\mathbb{R}^2$ if $c = 0$. Except for these strange cases, it turns out that it is indeed always a line.

Lemma 2.2.1 *Let L be a line with gradient m in $\mathbb{R}^2$, and suppose θ is the angle between L and the x-axis (measured anti-clockwise starting from the x-axis). Then, $m = \tan(\theta)$.*

Proof: (i) If $\theta = 0$, then L is the x-axis and $m = 0$. Because $\tan(0) = 0$, we do have $m = \tan(\theta)$.

(ii) Assume that $0 < \theta < \pi/2$. Let $(x_0, 0)$ be the point where L crosses the x-axis and let (x_1, y_1) be another point on L where $x_1 > x_0$, meaning it lies to the right of $(x_0, 0)$. We drop a perpendicular from (x_1, y_1) to the x-axis; it hits the x-axis at $(x_1, 0)$. We now draw this below.



By definition, the gradient is the quotient of the change in y by the change in x, which here is

$$m = \frac{y_1}{x_1 - x_0}.$$

Since the points (x_0, y_0), (x_1, y_1), $(x_1, 0)$ form a right-angled triangle, the definition of the tangent tells us that $\tan(\theta)$ is precisely the above fraction, that is $\tan(\theta) = m$.

(iii) Assume that $\pi/2 < \theta < \pi$. We can look at the supplement of θ, namely the angle $\Theta := \pi - \theta$ between the x-axis and L on the other side (which is acute). Proceeding like we did for (ii) above with Θ, we see that $\tan(\Theta) = -m$. However, since $\tan(\theta) = -\tan(\Theta)$, we get the result. $\square$

Note: We essentially only "sketched" the proof of (iii) above, meaning we didn't write all the details in full. This is perfectly fine, but you are encouraged to go through it with a fine-toothed comb to really understand what we are doing (and draw a picture to help).

Theorem 2.2.2 *Consider some line L in the plane $\mathbb{R}^2$.*

(i) *If it is non-vertical, contains (x_1, y_1) and has gradient m, then L satisfies*

$$y - y_1 = m(x - x_1).$$

(ii) *If it is non-vertical and contains distinct (x_1, y_1), (x_2, y_2), then $L\backslash\{(x_1, y_1)\}$ satisfies*

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

(iii) *If it contains $(p, 0)$ and $(0, q)$ for $p, q \neq 0$, then L satisfies*

$$\frac{x}{p} + \frac{y}{q} = 1.$$

Proof: (i) Using the $y = mx + c$ form, since L is assumed non-vertical, it can be described by $y_1 = mx_1 + c$, implying that the y-intercept $c = y_1 - mx_1$. Therefore, the equation of the line is $y = mx + y_1 - mx_1$; this easily rearranges to the desired formula.

(ii) Since L is non-vertical, it satisfies (i), i.e. it is described by the equation $y - y_1 = m(x - x_1)$. Therefore, assuming that $x \neq x_1$, we can divide this to get the following expression:

$$m = \frac{y - y_1}{x - x_1}.$$

Alternatively, we can write the gradient in terms of the two distinct points (x_1, y_1) and (x_2, y_2) as the change in y-coordinate over the change in x-coordinate, so equating this with our expression for m above gives the result.

(iii) Consider the general equation of a line $ax + by + c = 0$. We can substitute $(x, y) = (p, 0)$ and $(x, y) = (0, q)$ in turn into this equation. Doing so produces the following equalities:

$$ap + c = 0 \qquad \text{and} \qquad bq + c = 0.$$

We assumed that $p, q \neq 0$, so we can rearrange these to get $a = -c/p$ and $b = -c/q$. Hence, the equation of our line is $-cx/p - cy/q + c = 0$. We can factorise this to $-c(x/p + y/q - 1) = 0$. Because $c \neq 0$ (if it was, $(0, 0)$ would be on the line but the intercepts are assumed to not be at the origin), this final equation holds precisely when $x/p + y/q = 1$. □

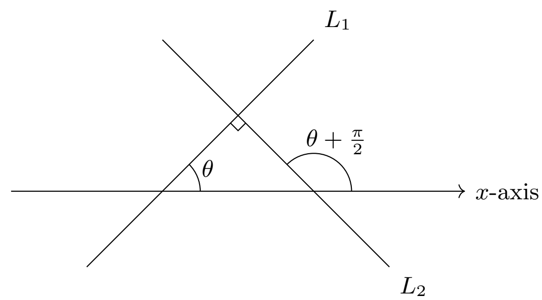
2.3 Perpendicular Lines

Theorem 2.3.1 *If L_1 and L_2 are perpendicular lines with gradients $m_1 \neq 0$ and m_2, then*

$$m_2 = -\frac{1}{m_1}.$$

Proof: Let θ be the angle between L_1 and the x-axis, measured anti-clockwise from the x-axis.

(i) Assume that $0 < \theta < \pi/2$. Because L_1 and L_2 are assumed perpendicular, this means rotating L_1 by $\pi/2$ about their point of intersection will transform it onto L_2. This is pictured below.



We may now apply Lemma 2.2.1 to these two lines to give us the below chain of equalities:

$$\begin{aligned}
m_2 &= \tan\big(\theta + \pi/2\big) \\
&= \frac{\sin\big(\theta + \pi/2\big)}{\cos\big(\theta + \pi/2\big)} \\
&= \frac{\sin(\theta)\cos\big(\pi/2\big) + \cos(\theta)\sin\big(\pi/2\big)}{\cos(\theta)\cos\big(\pi/2\big) - \sin(\theta)\sin\big(\pi/2\big)} \\
&= \frac{\cos(\theta)}{-\sin(\theta)} \\
&= -\frac{1}{\tan(\theta)} \\
&= -\frac{1}{m_1}.
\end{aligned}$$

(ii) Assume that $\pi/2 < \theta < \pi$. We proceed as above except with the supplement $\Theta := \pi - \theta$:

$$\begin{aligned}
m_2 &= \tan\big(\theta - \pi/2\big) \\
&= \tan\big(\pi - \Theta - \pi/2\big) \\
&= \tan\big(-\Theta + \pi/2\big) \\
&= \frac{\sin\big(-\Theta + \pi/2\big)}{\cos\big(-\Theta + \pi/2\big)} \\
&= \frac{\sin(-\Theta)\cos\big(\pi/2\big) + \cos(-\Theta)\sin\big(\pi/2\big)}{\cos(-\Theta)\cos\big(\pi/2\big) - \sin(-\Theta)\sin\big(\pi/2\big)} \\
&= \frac{\cos(-\Theta)}{-\sin(-\Theta)} \\
&= \frac{-\cos(\theta)}{\sin(\theta)} \\
&= -\frac{1}{\tan(\theta)} \\
&= -\frac{1}{m_1}.
\end{aligned}$$

□

Theorem 2.3.2 *If L_1 and L_2 be lines with gradients m_1 and m_2 such that $m_1 m_2 = -1$, then L_1 and L_2 are perpendicular.*

Proof: By assumption, we know that $m_2 = -1/m_1$. Appealing to Lemma 2.2.1, we can write

$$m_1 = \tan(\theta_1) \qquad \text{and} \qquad m_2 = -1/m_1 = \tan(\theta_2).$$

Combining these together produces $-1/\tan(\theta_1) = \tan(\theta_2)$ which means $1 - \tan(\theta_1)\tan(\theta_2) = 0$. From a compound-angle formula, we know that $\tan(\theta_1 + \theta_2)$ is undefined, but this can only be the case when $\theta_1 + \theta_2 = \pi/2$ (or an odd-multiple of it, but our angles are restricted to $0 < \theta_i < \pi$). Hence, the angle between L_1 and L_2 is a right angle, as required. □

2.4 Distance from a Point to a Line

Reminder: The distance from a point to a line is the length of the dropped perpendicular.

We now deduce a formula in Cartesian coordinates for the distance between a point and a line.

Proposition 2.4.1 *Let P be a point (x_0, y_0) not on the line L given by $ax + by + c = 0$, where a and b are* **not both** *zero. Then, the distance from P to L is given by*

$$\operatorname{dist}(P, L) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

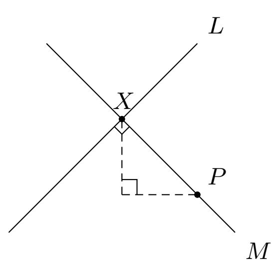
Proof: (i) Suppose $a, b \neq 0$ are both non-zero. Because L has gradient $-a/b$, it follows from Theorem 2.3.1 that the gradient of the line M which is perpendicular to L is b/a. Therefore, we can use Theorem 2.2.2(i) to write down the equation of M:

$$y - y_0 = \frac{b}{a}(x - x_0).$$

This can be rearranged into the usual form of an equation of a line, namely as

$$bx - ay + ay_0 - bx_0 = 0.$$

The aim of the game is to deduce the intersection point X, which in coordinates we call (x_1, y_1), of L and M. We then use Pythagoras' Theorem to get the formula we want. This is drawn here.



In order to find X, it amounts to solving the following equations simultaneously:

$$\left.\begin{aligned} ax_1 + by_1 + c &= 0 \\ bx_1 - ay_1 + ay_0 - bx_0 &= 0 \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} x_1 = \dfrac{b^2x_0 - aby_0 - ac}{a^2+b^2} \\ y_1 = \dfrac{a^2y_0 - abx_0 - bc}{a^2+b^2} \end{cases}.$$

We can work out the length of PX by using Pythagoras' Theorem, noting that the base of the triangle in the above picture has length $x_0 - x_1$ and the height has length $y_0 - y_1$. Consequently,

$$\begin{aligned} \operatorname{dist}(P, L) &= \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} \\ &= \sqrt{\left(\tfrac{a}{a^2+b^2}(ax_0 + by_0 + c)\right)^2 + \left(\tfrac{b}{a^2+b^2}(ax_0 + by_0 + c)\right)^2} \\ &= \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}(ax_0 + by_0 + c)^2} \\ &= \frac{1}{\sqrt{a^2+b^2}}\sqrt{(ax_0 + by_0 + c)^2} \\ &= \frac{1}{\sqrt{a^2+b^2}}|ax_0 + by_0 + c|. \end{aligned}$$

Note: The final equality above uses the fact that $\sqrt{z^2} = |z|$ for any real number $z \in \mathbb{R}$.

(ii) If $a = 0$, the line L is a horizontal line. Therefore, the distance between P and X will be the difference between the y-coordinate of P and any point lying on the line. One can easily verify this agrees with the formula in the statement.

(iii) If $b = 0$, the line L is a vertical line. Therefore, the distance between P and X will be the difference between the x-coordinate of P and any point lying on the line. One can easily verify this agrees with the formula in the statement. □

Method – Distance from a Point to a Line: Let L be a line and P a point not on L.

(i) Rewrite the equation of the line L in the form $ax + by + c = 0$.

(ii) Substitute a, b, c and the coordinates x_0, y_0 of point P into Proposition 2.4.1.

2.5 Circles

Definition 2.5.1 Let $\theta \in \mathbb{R}$ and, starting at the point $(1, 0) \in \mathbb{R}^2$, let P be the point arrived at by rotating $(1, 0)$ by angle θ about the origin. The convention is that the rotation is anti-clockwise for $\theta > 0$ and clockwise for $\theta < 0$. The cosine and sine of the angle is the x- and y-coordinate of P, respectively.

Note: We call such angles signed angles since they can be both positive and negative. In this way, we can sketch a picture of the situation described in Definition 2.5.1; see below.

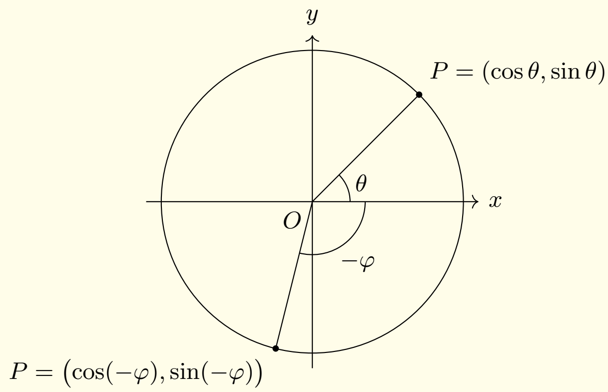


Figure 11: The sine and cosine of signed angles.

We will prove now that Definition 2.5.1 coincides with the one for triangles in the case $0 \leq \theta < \pi$.

Proposition 2.5.2 *The two definitions of* sin *and* cos *agree with each other.*

Proof: Let P be the point achieved by rotating $(1, 0)$ about the origin by θ, as in Figure 11. If $\theta = 0$ or $\theta = \pi/2$, this is clear. We therefore split into the acute and obtuse cases. But let's first establish some notation: denote by X the foot of the perpendicular dropped from P to the x-axis, and use $\sin_{\text{old}}(\theta)$ and $\cos_{\text{old}}(\theta)$ for the original definitions of sine and cosine, respectively.

(i) Assume that $0 < \theta < \pi/2$. Then, the old versions of cosine and sine are

$$\cos_{\text{old}}(\theta) = \frac{OX}{OP} = \frac{\cos(\theta)}{1} = \cos(\theta) \qquad \text{and} \qquad \sin_{\text{old}}(\theta) = \frac{PX}{OP} = \frac{\sin(\theta)}{1} = \sin(\theta)$$

(ii) Assume that $\pi/2 < \theta < \pi$. Then, for $\Theta \coloneqq \pi - \theta$, we have $P = (-\cos\Theta, \sin\Theta)$ and so

$$\cos_{\text{old}}(\theta) = -\cos(\Theta) = \cos(\theta) \qquad \text{and} \qquad \sin_{\text{old}}(\theta) = \sin(\Theta) = \sin(\theta).$$ □

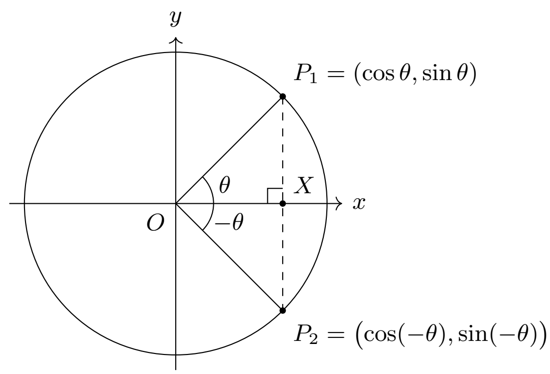
Theorem 2.5.3 *Let θ be a signed angle. Then, the following equalities hold:*

(i) $\sin(\theta + 2\pi) = \sin(\theta)$.
(ii) $\cos(\theta + 2\pi) = \cos(\theta)$.
(iii) $\sin(\pi - \theta) = \sin(\theta)$.
(iv) $\cos(\pi - \theta) = -\cos(\theta)$.
(v) $\sin(-\theta) = -\sin(\theta)$.
(vi) $\cos(-\theta) = \cos(\theta)$.
(vii) $\sin^2(\theta) + \cos^2(\theta) = 1$.
(viii) $\sin\big(\theta + \pi/2\big) = \cos(\theta)$.

Proof: (i)/(ii) These both follow from Definition 2.5.1, since if we wind around the origin by 2π, we end precisely at the point where we started.

(iii)/(iv) Suppose that $0 < \theta < \pi/2$. Then, this follows from the original definitions of sine and cosine for triangles (and therefore for the new definition by Proposition 2.5.2). We can write a similar argument for $0 \leq \theta < 2\pi$, and extend to $\theta \in \mathbb{R}$ via (i) and (ii).

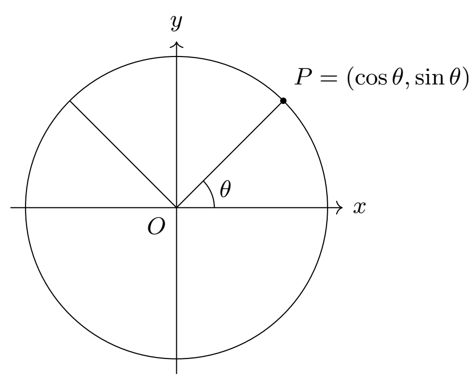
(v)/(vi) Suppose that $0 < \theta < \pi/2$. Then, we obtain two right-angled triangles drawn here.



Looking at the coordinates of P_2 in the right-angled triangle P_2OX, we get the equalities we want. We can write a similar argument for $0 \leq \theta < 2\pi$, and extend to $\theta \in \mathbb{R}$ via (i) and (ii).

(vii) This is immediate from Pythagoras' Theorem.

(viii) Suppose $0 < \theta < \pi/2$. Then, $\theta + \pi/2$



□

Definition A circle with centre $A = (a, b) \in \mathbb{R}^2$ and radius $r > 0$ is the set of points whose distance from A is precisely equal to r. We denote a circle by the calligraphic letter $\mathcal{C}$.

Proposition *A circle in* $\mathbb{R}^2$ *centred at* (a, b) *with radius* $r > 0$ *is given by the* locus equation

$$(x - a)^2 + (y - b)^2 = r^2.$$

Proof: Let $P = (x, y) \in \mathbb{R}^2$ be any point in the Cartesian plane. Then, Pythagoras' Theorem tells us that the squared distance from the circle's centre $A = (a, b)$ is

$$AP^2 = (x - a)^2 + (y - b)^2.$$

Hence, as every point on a circle is exactly distance r from A, we obtain the result. □

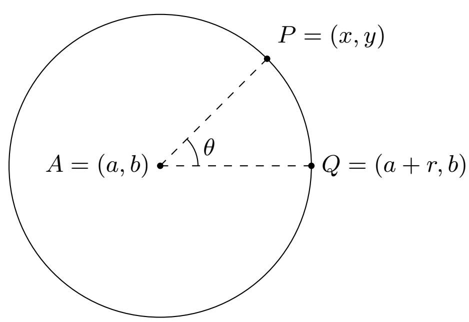
Proposition 2.5.4 *A point* (x, y) *lies on the circle in* $\mathbb{R}^2$ *centred at* (a, b) *with radius* $r > 0$ *if and only if there exists some number* θ *where* $0 \leq \theta < 2\pi$ *such that*

$$(x, y) = (a + r\cos\theta, b + r\sin\theta).$$

Proof: ($\Leftarrow$) Suppose that such θ exists and (x, y) has the given form. Then,

$$\begin{aligned}(x - a)^2 + (y - b)^2 &= (a + r\cos\theta - a)^2 + (b + r\sin\theta - b)^2 \\ &= r^2\cos^2(\theta) + r^2\sin^2(\theta) \\ &= r^2(\cos^2\theta + \sin^2\theta) \\ &= r^2.\end{aligned}$$

($\Rightarrow$) Suppose that $P = (x, y)$ lies on the circle, and let Q be the point on the circle with the same y-coordinate as the centre $A = (a, b)$, that is $Q = (a + r, b)$. Then, the radius AP is obtained from AQ by rotating through a signed angle $0 \leq \theta < 2\pi$. The picture when θ is acute is this.



From the triangle definition of cosine and sine, we have $\cos(\theta) = (x - a)/r$ and $\sin(\theta) = (y - b)/r$, which rearrange to $x = a + r\cos(\theta)$ and $y = b + r\sin(\theta)$ respectively. A similar argument works for other values of θ (by using the facts established in Theorem 2.5.3). □

Note: The expression in Proposition 2.5.4 is known as the parametric equation of a circle.

3 Conic Sections

3.1 Introduction

Definition 3.1.1 A conic (or conic section) is the locus of a point in the plane whose distance from a fixed point is a positive constant multiple of its distance from a fixed line not containing said point.

Don't worry if this doesn't paint a nice picture in your mind; we will draw one now. Indeed, let P be a point on some conic. Then, label the fixed point referred to above F (called the focus), the positive constant $e \in \mathbb{R}^+$ (called the eccentricity) and the fixed line L (called the directrix).

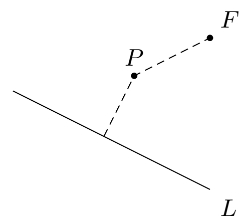


Figure 12: A point P on a conic with focus F and directrix L.

Note: The equation to have in mind with this description of a conic is $PF = e\,\mathrm{dist}(P, L)$.

The goal is to describe these conics using Cartesian coordinates. We will do this by considering different values of the eccentricity and studying the resulting conics. The values are as follows:

$$0 < e < 1, \qquad e = 1, \qquad e > 1.$$

Remark We could allow $e = 0$ b replacing "positive constant multiple" in Definition 3.1.1 with "*non-negative* constant multiple" to cover this situation. However, it turns out we have already looked at this case before: when $e = 0$, the conic in question is nothing more than a circle!

3.2 The Parabola

Definition The parabola is the conic we obtain by setting $e = 1$, meaning $PF = \mathrm{dist}(P, L)$.

Here, we choose a coordinate system suitable for the parabola, namely the following:

- The origin O will be halfway between F and L.
- The x-axis will be the line through F and O, perpendicular to L.
- The y-axis will be the line through O, parallel to L.

Definition 3.2.1 The standard equation of the parabola is $y^2 = 4ax$ where $a \neq 0$.

Remark We can derive the above equation by using the fact $PF = \text{dist}(P, L)$ and interpreting this equality in Cartesian coordinates. Indeed, let $F = (a, 0)$, meaning the directrix L is of the form $x = -a$ (in the coordinate system we established just above). Then,

$$\begin{aligned} PF = \text{dist}(P, L) &\Leftrightarrow \sqrt{(x-a)^2 + y^2} = |x + a| \\ &\Leftrightarrow x^2 - 2ax + a^2 + y^2 = (x+a)^2 \\ &\Leftrightarrow y^2 = 4ax. \end{aligned}$$

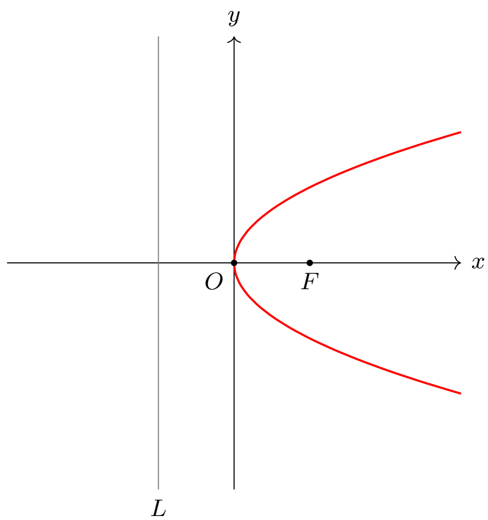


Figure 13: The standard parabola $y^2 = 4ax$ with its focus and directrix.

Note: If we change $a \neq 0$, we obtain parabolas of slightly different shapes than that in Figure 13. In particular, if a is smaller, the parabola looks vertically squashed. Of course, if a is negative, then the parabola would face the other way (reflected across the y-axis).

3.3 Change of Coordinates

If a conic is given by a locus equation, the choice of axes might not be as convenient as we have above for the parabola. We may want to alter the coordinates so that they do become simpler.

Definition A translation from (x, y) coordinates to (X, Y) coordinates is a transformation

$$\begin{aligned} x &= X + c \\ y &= Y + d \end{aligned} \qquad \text{for any } c, d \in \mathbb{R}.$$

The motivation is this: (c, d) in (x, y) coordinates becomes $(0, 0)$ in the new (X, Y) coordinates.

Remark We picture the translation in Figure 14 below. Note that other so-called *rigid transformations* (ones that preserve distances) are possible, such as rotations and reflections, as well as combinations of these with translations.

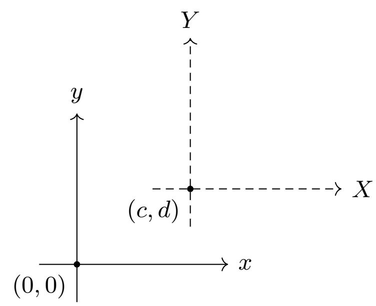


Figure 14: Translation from (x, y) coordinates to (X, Y) coordinates.

Method – Sketching a Parabola: Assume we have a locus equation of a parabola.

(i) Complete the square in y to obtain something like $(y+d)^2 = 4a(x+c)$.
(ii) Apply the translation change of coordinates $X = x + c$ and $Y = y + d$.
(iii) Sketch the standard parabola $Y^2 = 4aX$ and determine its focus and directrix.
(iv) Reverse the change of coordinates to get everything in the original (x, y) system.

3.5 Parametric Form of the Parabola

Theorem 3.5.1 *Let $a \neq 0$. Then, a point $P = (x, y)$ in the plane lies on the standard parabola $y^2 = 4ax$ if and only if there exists $t \in \mathbb{R}$ such that $P = (at^2, 2at)$.*

Proof: ($\Rightarrow$) Given $y^2 = 4ax$, take $t = y/2a$. Hence, $y = 2at$ and $x = y^2/4a = 4a^2t^2/4a = at^2$.

($\Leftarrow$) Given $(x, y) = (at^2, 2at)$ for some $t \in \mathbb{R}$, we have $y^2 = 4a^2t^2 = 4a(at^2) = 4ax$, as needed. $\square$

Proposition *Consider the standard parabola $y^2 = 4ax$ with $a \neq 0$ and let $P = (ap^2, 2ap)$ be a point on it where $p \neq 0$. Then, the tangent to the parabola at P is given by*

$$y = \frac{1}{p}x + ap.$$

Proof: As is the usual story, we differentiate (implicitly, here) in order to find the gradient:

$$\frac{\mathrm{d}}{\mathrm{d}x}[y^2] = \frac{\mathrm{d}}{\mathrm{d}x}[4ax] \qquad \Rightarrow \qquad 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a \qquad \Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}.$$

We substitute the coordinates of P into the above derivative to conclude the gradient of the tangent line is $1/p$. Consequently, we use Theorem 2.2.2(i) to determine the tangent's equation:

$$y - y_1 = m(x - x_1) \qquad \Rightarrow \qquad y - 2ap = \frac{1}{p}(x - ap^2) \qquad \Rightarrow \qquad py = x + ap^2.$$

Finally, we divide through by p (recall we assume $p \neq 0$) to get the expression we are after. $\square$

Remark Where does the name "focus" come from? Well then, it turns out that parabolas reflect parallel lines to a point (they *focus* them). It is this principle why mirrors that form part of the headlight of a car are parabolic. To prove this mathematically, take any line parallel K to the x-axis and suppose it intersects the parabola at point $P = (ap^2, 2ap)$. The angle θ between the line K and the tangent to the parabola at P is given by

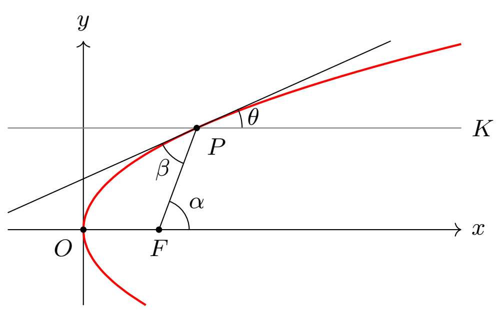
$$\tan(\theta) = \frac{1}{p},$$

where $1/p$ we recall is the gradient of the tangent line. The angle α between PF and the x-axis satisfies a similar condition, namely taking the tan of the angle is the gradient of the line:

$$\tan(\alpha) = \frac{2ap}{ap^2 - a} = \frac{2p}{p^2 - 1} = \frac{2/p}{1 - 1/p^2} = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} = \tan(2\theta).$$

Because $0 \leq \alpha \leq \pi$ and $0 \leq 2\theta \leq \pi$, we conclude from above that $\alpha = 2\theta$. Next, suppose β is the angle between PF and the tangent line to the parabola at P. We can now use the fact that vertically opposite angles are equal (Proposition 1.7.5) to conclude

$$\alpha = \beta + \theta \qquad \Rightarrow \qquad \beta = \theta.$$



The picture above frames all this mathematics geometrically. But the point is that our line K was completely arbitrary, so in fact all parallel lines reflect from the parabola at the same angle.

Definition 3.5.3 A chord of a parabola is a line segment between two distinct points on it.

Proposition 3.5.4 *Let $y^2 = 4ax$ be the standard parabola containing two distinct points $P = (ap^2, 2ap)$ and $Q = (aq^2, 2aq)$ for $p, q \in \mathbb{R}$ and let L be the line through P and Q.*

(i) *If $p + q \neq 0$, then L has gradient $\dfrac{2}{p+q}$.*

(ii) *The equation of L is $(p+q)y = 2x + 2apq$.*

Proof: (i) Since $p \pm q \neq 0$, by assumption and because P and Q are distinct, the gradient of L is

$$\frac{2ap - 2aq}{ap^2 - aq^2} = \frac{2(p-q)}{(p-q)(p+q)} = \frac{2}{p+q}.$$

(ii) If we use Theorem 2.2.2(i) along with the gradient from above, the equation of L is

$$y-2ap=\frac{2}{p+q}(x-ap^2) \qquad \Leftrightarrow \qquad (p+q)(y-2ap)=2(x-ap^2) \qquad \Leftrightarrow \qquad (p+q)y=2x+2apq.$$

On the other hand, if $p+q=0$ (note we do not omit this possibility in the statement of the proposition for this part), then $P=(ap^2,2ap)$ and $Q=(ap^2,-2ap)$. This tells us the line L is vertical (since it passes through both P and Q) and is given by $x=ap^2$. But this is equivalent to $(p+q)y=2x+2apq$ is we substitute $p=-q$. □

3.6 The Ellipse

Definition The ellipse is the conic we obtain by setting $0<e<1$.

Lemma 3.6.1 *Consider a conic with eccentricity $e\neq 1$. It is possible to make a change of variables from (x,y)-coordinates to (X,Y)-coordinates such that the focus F becomes the origin and the directrix L is parallel to the Y-axis, that is of the form $X=-h$ for some $h\in\mathbb{R}$. Then, the conic itself satisfies*

$$\left(X-\frac{he^2}{1-e^2}\right)^2+\frac{Y^2}{1-e^2}=\frac{h^2e^2}{(1-e^2)^2}.$$

Proof: Again, we use the earlier note which tells us a point $P=(X,Y)$ lies on a conic if and only if it satisfies $PF=e\,\mathrm{dist}(P,L)$. All we need do is apply the relevant distance formulae and tweak things with a bit of algebraic manipulation:

$$\begin{aligned}
\mathrm{dist}(P,F)=e\,\mathrm{dist}(P,L) \quad &\Leftrightarrow \quad \sqrt{X^2+Y^2}=e|X+h| \\
&\Leftrightarrow \quad X^2+Y^2=e^2(X^2+2Xh+h^2) \\
&\Leftrightarrow \quad X^2(1-e^2)+Y^2-2Xhe^2=h^2e^2 \\
&\Leftrightarrow \quad X^2+\frac{Y^2}{1-e^2}-\frac{2Xhe^2}{1-e^2}=\frac{h^2e^2}{1-e^2} \\
&\Leftrightarrow \quad \left(X-\frac{he^2}{1-e^2}\right)-\frac{h^2e^4}{(1-e^2)^2}+\frac{Y^2}{1-e^2}=\frac{h^2e^2}{1-e^2} \\
&\Leftrightarrow \quad \left(X-\frac{he^2}{1-e^2}\right)+\frac{Y^2}{1-e^2}=\frac{h^2e^4}{(1-e^2)^2}+\frac{h^2e^2}{1-e^2} \\
&\Leftrightarrow \quad \left(X-\frac{he^2}{1-e^2}\right)+\frac{Y^2}{1-e^2}=\frac{he^2}{(1-e^2)^2}.
\end{aligned}$$
□

Note: The only assumption in Lemma 3.6.1 is $e\neq 1$, so it will apply also when $e>1$.

Theorem 3.6.2 *There exist axes in the coordinates (x, y) and real numbers $a > b > 0$ such that the equation of the ellipse is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Proof: Because the eccentricity $0 < e < 1$, we can use Lemma 3.6.1 to conclude that there are coordinates (X, Y) such that the ellipse has equation

$$\left(X - \frac{he^2}{1-e^2}\right)^2 + \frac{Y^2}{1-e^2} = \frac{h^2e^2}{(1-e^2)^2}$$

and the directrix is given by $X = -h$. If we set $a := he/(1-e^2)$, the above equation becomes

$$(X - ae)^2 + \frac{Y^2}{1-e^2} = a^2 \qquad \Leftrightarrow \qquad \frac{(X-ae)^2}{a^2} + \frac{Y^2}{a^2(1-e^2)} = 1.$$

Translating to new coordinates $x := X - ae$, $y := Y$ and set $b := \sqrt{a^2(1-e^2)}$, the above becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If $a < 0$, we can replace it with $-a$ to ensure it is positive. Then, $b > 0$ and we again have that $b < a$ since $0 < e < 1$ so the equation still holds. □

Theorem 3.6.3 *Let $a > b > 0$ be real numbers. Then, the curve defined by the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is an ellipse with eccentricity $e = \sqrt{1 - b^2/a^2}$, foci $(\pm ae, 0)$ and directrices $x = \pm a/e$.

Proof: Since $a > b$, let $e := \sqrt{1 - b^2/a^2}$. Then, $b^2 = a^2(1-e^2)$ so we re-write the equation as

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

To show this is a conic (specifically an ellipse), we wish to write it in the form $PF = e\,\mathrm{dist}(P, L)$. This will also allow us to determine the foci and directrices. Indeed,

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \qquad &\Leftrightarrow \qquad x^2(1-e^2) + y^2 = a^2(1-e^2) \\
&\Leftrightarrow \qquad x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2 \\
&\Leftrightarrow \qquad (x-ae)^2 + y^2 = e^2(x - a/e)^2 \\
&\Leftrightarrow \qquad \sqrt{(x-ae)^2 + y^2} = e\big|x - a/e\big|.
\end{aligned}$$

The above is precisely the form we want: it says that the distance between (x, y) and $(ae, 0)$ is equal to e multiplied by the distance from (x, y) to the line $x = a/e$. Note that the second line comes from expanding, rearranging and artificially subtracting $2aex$ from both sides so that it has a nice factorisation. On the other hand, we could **add** $2aex$ to both sides to obtain the following alternate equation:

$$\sqrt{(x+ae)^2 + y^2} = e\big|x + a/e\big|.$$

Hence, the foci are $(\pm ae, 0)$ and the directrices are $x = \pm a/e$ as we expected. $\square$

Definition 3.6.4 The standard equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$.

- The centre is the point $(0, 0)$.
- The vertices are the points $(a, 0)$ and $(-a, 0)$.
- The major axis is the line from $(-a, 0)$ to $(a, 0)$.
- The minor axis is the line from $(0, -b)$ to $(0, b)$.
- The semi-axis lengths are a and b.

Remark 3.6.5 The standard ellipse is symmetric about the x- and y-axes; note that reflecting in either of them sends foci to foci and directrices to directrices. Furthermore, since $a > b$, the ellipse is wider than it is high (as is clear from Figure 15). Note that it also lies in the rectangle of width $2a$ and height $2b$, that is it is a subset of $\{(x, y) \in \mathbb{R}^2 : |x| \leq a \text{ and } |y| \leq b\}$.

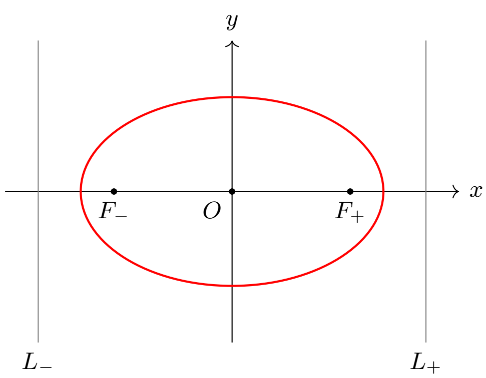


Figure 15: The standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its foci and directrices.

Note: A circle can be thought of as the degenerate ellipse where $a = b$. Be aware

Method – Sketching an Ellipse: Assume we have a locus equation of an ellipse.

(i) Complete the square and rearrange to obtain something like $\frac{(x+c)^2}{a^2} + \frac{(y+d)^2}{b^2} = 1$.
(ii) Apply the translation change of coordinates $X = x + c$ and $Y = y + d$.
(iii) Sketch the standard ellipse $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$ and determine its foci and directrices.
(iv) Reverse the change of coordinates to get everything in the original (x, y) system.

3.7 Parametric Form of the Ellipse

Theorem 3.7.1 *Let $a > b > 0$. Then, a point $P = (x, y)$ in the plane lies on the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if and only if there exists $0 \leq t < 2\pi$ such that $P = (a \cos t, b \sin t)$.*

Proof: ($\Rightarrow$) Suppose (x, y) satisfies the equation and consider new coordinates (X, Y) given by

$$X = x \qquad \text{and} \qquad Y = \frac{a}{b}y.$$

The equation we have then becomes $X^2 + Y^2 = a^2$, meaning that (X, Y) lies on a circle centred at O with radius a. Therefore, we know from Proposition 2.5.4 that there exists $t \in \mathbb{R}$ with $0 \leq t < 2\pi$ such that $(X, Y) = (a \cos t, a \sin t)$. If we revert back to the (x, y) coordinates, this becomes $(x, y) = (a \cos t, b \sin t)$ as expected.

($\Leftarrow$) Suppose $(x, y) = (a \cos t, b \sin t)$. Then, substituting into the equation tells us

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2(t) + \sin^2(t) = 1.$$ □

Note: The change of variables for the "only if" direction of Theorem 3.7.1 is **not** a rigid motion: because it changes the ellipse to a circle, it clearly doesn't preserve distances. However, this argument is valid because we reversed the transformation at the end.

Lemma 3.11.2 *Let $a > b > 0$. If P is a point on the standard ellipse with foci $F_\pm$, then*

$$PF_+ + PF_- = 2a.$$

Sketch of Proof: We can use the parametric form of the ellipse along with Pythagoras' Theorem to see that $PF_\pm = a(\mp e \cos t + 1)$. We have to be careful when square-rooting here: since $e < 1$ and $-1 \leq \cos(t) \leq 1$, we take the negative root for PF_+ and the positive root for PF_-. Hence, it is a straightforward observation that $PF_+ + PF_- = -ae\cosh(t) + a + ae\cosh(t) + a = 2a$. □

3.9 The Hyperbola

Definition The hyperbola is the conic we obtain by setting $e > 1$.

Theorem 3.9.1 *Let $a, b > 0$ be real numbers. Then, the curve defined by the equation*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is aa hyperbola with eccentricity $e = \sqrt{1 + b^2/a^2}$, foci $(\pm ae, 0)$ and directrices $x = \pm a/e$.

Proof: Since $a, b > 0$, let $e \coloneqq \sqrt{1 + b^2/a^2}$. Then, $b^2 = a^2(e^2 - 1)$ so we re-write the equation as

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1.$$

To show that this is a conic (specifically a hyperbola), we wish to write it as $PF = e\,\mathrm{dist}(P, L)$. This will also allow us to determine the foci and directrices. Indeed,

$$\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \quad &\Leftrightarrow \quad x^2(e^2-1) - y^2 = a^2(e^2-1) \\
&\Leftrightarrow \quad x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2 \\
&\Leftrightarrow \quad (x - ae)^2 + y^2 = e^2(x - a/e)^2 \\
&\Leftrightarrow \quad \sqrt{(x-ae)^2 + y^2} = e\big|x - a/e\big|.
\end{aligned}$$

The above is precisely the form we want: it says that the distance between (x, y) and $(ae, 0)$ is equal to e multiplied by the distance from (x, y) to the line $x = a/e$. Note that the second line comes from expanding, rearranging and artificially subtracting $2aex$ from both sides so that it has a nice factorisation. On the other hand, we could **add** $2aex$ to both sides to obtain the following alternate equation:

$$\sqrt{(x+ae)^2 + y^2} = e\big|x + a/e\big|.$$

Hence, the foci are $(\pm ae, 0)$ and the directrices are $x = \pm a/e$ as we expected. □

Definition 3.9.2 The standard equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a, b > 0$.

- The centre is the point $(0, 0)$.
- The vertices are the points $(a, 0)$ and $(-a, 0)$.

Reminder: An asymptote of a curve is a line such that the distance between the curve and the line approaches zero as one (or both) of the coordinates tends to infinity.

Remark 3.9.5 We can factorise the left-hand side of the standard hyperbola as follows:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \Leftrightarrow \quad \left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 1.$$

We see that if x and y are both large and have the same sign, then the second factor $x/a + y/b$ is also very large, meaning that the first factor $x/a - y/b$ must be very small. Therefore, the point (x, y) in this case is close to the line $x/a - y/b = 0$. On the other hand, if they are both large with **opposite** signs, then the opposite argument works and (x, y) is close to the line $x/a + y/b = 0$.

Definition 3.9.6 The asymptotes of the standard hyperbola are the lines given by $\frac{x}{a} \pm \frac{y}{b} = 0$.

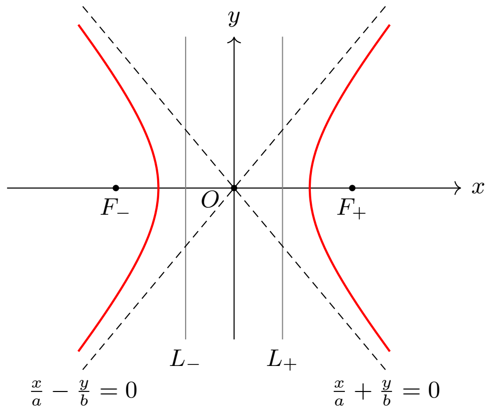


Figure 16: The standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci, directrices and asymptotes.

Method – Sketching a Hyperbola: Assume we have a locus equation of a hyperbola.

(i) Complete the square and rearrange to obtain something like $\frac{(x+c)^2}{a^2} - \frac{(y+d)^2}{b^2} = 1$.

(ii) Apply the translation change of coordinates $X = x + c$ and $Y = y + d$.

(iii) Sketch the standard ellipse $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$ and determine its foci and directrices.

(iv) Reverse the change of coordinates to get everything in the original (x, y) system.

3.11 Parametric Form of the Hyperbola

Theorem 3.11.1 *Let $a, b > 0$. Then, a point $P = (x, y)$ in the plane lies on the standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if and only if one of the following occurs:*

(i) *There exists $-\pi \le t < \pi$ with $t \ne \pi/2$ such that $P = (a \sec t, b \tan t)$.*

(ii) *There exists $t \in \mathbb{R}$ such that $(x, y) = (a \cosh t, b \sinh t)$.*

Proof: Omitted; this is very similar to the proofs of Theorems 3.5.1 and 3.7.1. □

Note: The parametrisation in Theorem 3.11.1(ii) only gives the right-hand branch of the hyperbola. Of course, since the standard hyperbola is symmetric about the y-axis, we can cover both by saying that P is given by precisely one of $(x, y) = (\pm a \cosh t, b \sinh t)$.

Remark We can use the second parametric form of the hyperbola to give an alternate definition of the hyperbola in terms of distances from a point on the hyperbola to each foci, namely

$$PF_+ - PF_- = -2a,$$

where $F_\pm$ are the foci. Indeed, we use Pythagoras' Theorem to see that $PF_\pm = a(e \cosh t \mp 1)$. This requires the fact that $e > 1$ and $\cosh(t) \ge 1$; this allows us to take only the positive square root. It is now an easy observation that $PF_+ - PF_- = ae\cosh(t) - a - ae\cosh(t) + (-a) = -2a$.

3.12 Summary

We now provide a summary of the three conics we have introduced and their important properties.

	Parabola	**Ellipse**	**Hyperbola**
Eccentricity Value	$e = 1$	$0 < e < 1$	$e > 1$
Equation	$y^2 = 4ax$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Restrictions	$a \neq 0$	$a > b > 0$	$a, b > 0$
Centre	—	$(0, 0)$	$(0, 0)$
Vertices	$(0, 0)$	$(\pm a, 0)$	$(\pm a, 0)$
Foci	$(a, 0)$	$(\pm ae, 0)$	$(\pm ae, 0)$
Directrices	$x = -a$	$x = \pm\frac{a}{e}$	$x = \pm\frac{a}{e}$
Asymptotes	—	—	$\frac{x}{a} \pm \frac{y}{b} = 0$
Eccentricity Formula	—	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 + \frac{b^2}{a^2}}$

Table 1: A summary of the important properties of each conic.

Remark The name "conic" comes from the following interpretation: consider a cone and intersect it with a plane. The orientation of the plane will change what the intersection looks like: the circle is contained on a horizontal plane; the ellipse is contained on an angled plane through both sides of the cone; the parabola is contained on an angled plane through one side and the base of the cone; the hyperbola is contained on a vertical plane. We can see that the hyperbola is the only conic with more than a single one piece. This is all pictured in Figure 17 below.

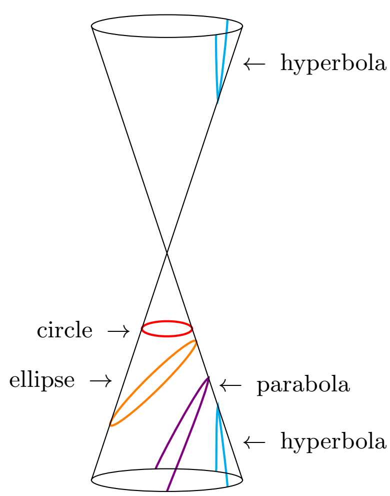


Figure 17: The conics arising from the intersection of a cone with various planes.

4 Polar Coordinates and Axis Rotation

4.1 Introduction

A common alternative to the Cartesian coordinate system is the so called *polar* system, which is now introduced. We will soon show that there is a relatively hassle-free way to convert between the Cartesian and polar systems.

Definition The polar coordinate system consists of the following choices:

- Choose a pole, O.
- Choose a polar axis X through O.
- Choose a scale on the polar axis.

In this system, a point P is specified by the distance r away from the pole and the angle θ between the line segment PO and the polar axis. By convention, we measure the angle anti-clockwise from X (regarded as a signed angle, so a negative corresponds to a clockwise measurement).

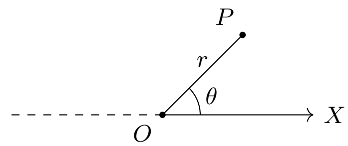


Figure 18: The plane as described by polar coordinates.

Definition 4.1.1 The polar coordinates of a point P are $[r, \theta]$ where $r \geq 0$ and $0 \leq \theta < 2\pi$.

Notation We use square brackets for polar coordinates to distinguish from Cartesian coordinates.

Note: The point $(0, 0)$ is given by $[0, \theta]$ for any θ in polar coordinates (in particular, $[0, 0]$).

Remark 4.1.3 We often impose (and did so in Definition 4.1.1) $r \geq 0$. However, we can allow $r < 0$ because this just means that our point $[r, \theta]$ is distance $|r|$ away from the pole O at an angle of $\theta + \pi$ still measured anti-clockwise from X.

4.2 Converting Between Polar and Cartesian Coordinates

Lemma 4.2.1 *If P has polar coordinates $[r, \theta]$, it has Cartesian coordinates (x, y) with*

$$x = r\cos(\theta) \qquad \textit{and} \qquad y = r\sin(\theta).$$

Proof: All the way from Definition 2.5.1, the point arrived at by doing an anti-clockwise rotation of $(1, 0)$ about the origin by angle θ has coordinates $(\cos\theta, \sin\theta)$. Since P is arrived at by rotating $(r, 0)$ in this way, the coordinates are simply $(r\cos\theta, r\sin\theta)$. $\square$

Reminder: In general, $\tan^{-1}(x)$ isn't well-defined since if $\tan(\theta) = x$, then $\tan(\theta + n\pi) = x$ for all $n \in \mathbb{Z}$. Therefore, we restrict to the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to compute $\tan^{-1}(x)$.

Lemma 4.2.2 *If P has Cartesian coordinates (x, y), it has polar coordinates $[r, \theta]$ with*

$$r = \sqrt{x^2 + y^2} \qquad \textit{and} \qquad \theta = \begin{cases} \tan^{-1}(y/x), & \textit{if } x > 0 \\ \tan^{-1}(y/x) + \pi, & \textit{if } x < 0 \\ \pi/2, & \textit{if } x = 0 \textit{ and } y > 0 \\ -\pi/2, & \textit{if } x = 0 \textit{ and } y < 0 \\ \textit{undefined}, & \textit{if } x = 0 \textit{ and } y = 0 \end{cases}.$$

Proof: There are a number of cases to consider.

(i) If P is in quadrant one ($x > 0, y > 0$), then $0 \leq \theta < \frac{\pi}{2}$ which is simply $\tan^{-1}(y/x)$.

(ii) If P is in quadrant two ($x < 0, y > 0$), then $\frac{\pi}{2} < \theta \leq \pi$. We can now define an acute angle $\alpha = \pi - \theta$. One needs to substitute $\alpha = \tan^{-1}(y/-x) = -\tan^{-1}(y/x)$ and rearrange.

(iii) If P is in quadrant three ($x < 0, y < 0$), then $\pi \leq \theta < \frac{3\pi}{2}$. We can now define an acute angle $\alpha = \theta - \pi$. One needs to substitute $\alpha = \tan^{-1}(-y/-x) = \tan^{-1}(y/x)$ and rearrange.

(iv) If P is in quadrant four ($x > 0, y < 0$), then $\frac{3\pi}{2} < \theta \leq 2\pi$. We can now define an acute angle $\alpha = 2\pi - \theta$. One needs to substitute $\alpha = \tan^{-1}(-y/x) = -\tan^{-1}(y/x)$ and rearrange.

Now, $\tan^{-1}(y/x)$ is undefined when $x = 0$, meaning $\theta = \pm\pi/2$ depending on the orientation. Finally, the fact that $r = \sqrt{x^2 + y^2}$ is a straightforward application of Pythagoras' Theorem. □

Note: An alternate way to find θ from the Cartesian coordinates is to first find r and solve the system of simultaneous equations $\cos(\theta) = x/r$ and $\sin(\theta) = y/r$.

4.3 Rotation of Axes

Given a Cartesian coordinate system (x, y), it is possible to rotate the axes anti-clockwise through a (signed) angle to obtain new axes (X, Y), giving us a new coordinate system. We can use polar coordinates to describe this.

Theorem 4.3.1 *Let (x, y) be Cartesian coordinates and (X, Y) the coordinates obtained by rotating the x- and y-axes through an angle α. Then, the new coordinates are*

$$X = x\cos(\alpha) + y\sin(\alpha) \qquad \textit{and} \qquad Y = -x\sin(\alpha) + y\cos(\alpha).$$

Proof: Let P be a point in the plane with Cartesian coordinates (x, y) and polar coordinates $[r, \theta]$. Suppose that $[R, \Theta]$ are the polar coordinates of P where we take the polar axis to be the

X-axis. Then, $R = r$ and $\Theta = \theta - \alpha$. With this information, we can use some compound-angle formulae with Lemma 4.2.1 to get the desired result:

$$\begin{aligned} X &= R\cos(\Theta) \\ &= r\cos(\theta - \alpha) \\ &= r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\ &= x\cos(\alpha) + y\sin(\alpha) \end{aligned}$$

and

$$\begin{aligned} Y &= R\sin(\Theta) \\ &= r\sin(\theta - \alpha) \\ &= r(\sin\theta\cos\alpha - \cos\theta\sin\alpha) \\ &= y\cos(\alpha) - x\sin(\alpha). \end{aligned}$$

□

Corollary 4.3.2 *Let (x, y) be Cartesian coordinates and (X, Y) the coordinates obtained by rotating the x- and y-axes through an angle α. Then, the original coordinates are*

$$x = X\cos(\alpha) - Y\sin(\alpha) \qquad \textit{and} \qquad y = X\sin(\alpha) + Y\cos(\alpha).$$

Proof: We can use Theorem 4.3.1 in the context of starting with (X, Y) and rotating by $-\alpha$ to get to (x, y); this would require us to interchange the lowercase and uppercase letters in the theorem. Substituting $-\alpha$ would then produce precisely the expressions we are after, since $\cos(-\alpha) = \cos(\alpha)$ and $\sin(-\alpha) = -\sin(\alpha)$. □

Reminder: An $m \times n$ matrix is an array of numbers that has m rows and n columns.

Remark 4.3.3 In matrix language, we can write Theorem 4.3.1 rather succinctly as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We call the 2×2 matrix above the *rotation matrix* and it has *determinant* (whatever that means) $\cos^2 + \sin^2 = 1$. Therefore, we can invert it; this is precisely what Corollary 4.3.2 says:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Note: We will **not** be using the matrix approach here, so ignore Remark 4.3.3 if you wish.

Remark 4.3.5 Rotating a shape through an angle of $-\alpha$ while keeping the axes fixed is exactly the same as keeping the shape fixed while rotating the axes through an angle of α.

4.4 Eliminating the xy-Term

Throughout, consider the following general equation in two variables where $A, B, C, D, E, F \in \mathbb{R}$:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \qquad (*)$$

Lemma 4.4.1 *Consider the curve defined by* $(*)$ *where* $A \neq C$ *and let* α *be defined by*

$$\tan(2\alpha) = \frac{B}{A - C}.$$

Then, after rotating the axes by α*, the equation in the new coordinates is of the form*

$$aX^2 + cY^2 + dX + eY + f = 0,$$

where $a, c, d, e, f \in \mathbb{R}$ *are possibly different from* $A, C, D, E, F \in \mathbb{R}$ *respectively.*

Proof: Recall the rotation of axes formulae from Corollary 4.3.2, which apply here:

$$x = X\cos(\alpha) - Y\sin(\alpha) \qquad \text{and} \qquad y = X\sin(\alpha) + Y\cos(\alpha).$$

If we focus only on the second-order terms in the curve defined by $(*)$, namely $Ax^2 + Bxy + Cy^2$. Note that because we wish to eliminate xy, we don't need to consider lower-order terms; of course they are affected but for the purposes of the proof, they play no role. Using the rotation of axes and expanding, we obtain the following expression where we don't care about the coefficients of X^2 and Y^2 (hence why we label them a and c respectively):

$$Ax^2 + Bxy + Cy^2 = aX^2 + \Big(-2A\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha) + 2C\cos\alpha\sin\alpha\Big)XY + cY^2.$$

This tells us that in order to eliminate the XY-term, we must choose the angle α such that

$$\begin{aligned}
& -2A\sin(\alpha)\cos(\alpha) + B(\cos^2\alpha - \sin^2\alpha) + 2C\cos(\alpha)\sin(\alpha) = 0 \\
\Rightarrow \quad & 2(C - A)\sin(\alpha)\cos(\alpha) + B(\cos^2\alpha - \sin^2\alpha) = 0 \\
\Rightarrow \quad & (C - A)\sin(2\alpha) + B\cos(2\alpha) = 0 \\
\Rightarrow \quad & \frac{\sin(2\alpha)}{\cos(2\alpha)} = \frac{B}{A - C},
\end{aligned}$$

as we want. However, in the case that $\cos(2\alpha) = 0$, we have $\sin(2\alpha) \neq 0$. Because we assume that $A \neq C$, there are no solutions with $\cos(2\alpha)$ and thus we can eliminate the XY-term again by choosing α such that the above is satisfied. ☐

Note: If $A = C$ in $(*)$, we can eliminate the xy-term simply by a rotation through $\alpha = \pi/4$.

5 Curves in Polar Coordinates

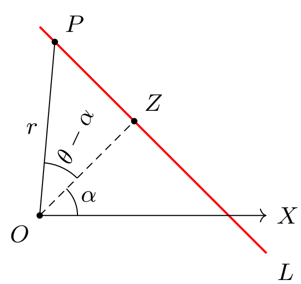
Note: We use the convention that $r < 0$ means we rotate by an angle of $\theta + \pi$ anti-clockwise.

5.1 Straight Lines in Polar Coordinates

Theorem 5.1.1 *Let L be a line at distance $d > 0$ from O and suppose the perpendicular from O to L meets the polar line at an angle α. Then, the equation of L is*

$$r\cos(\theta - \alpha) = d.$$

Proof: Let Z be the foot of the perpendicular from O to L. If P is an arbitrary point on the line L with polar coordinates $[r, \theta]$, then the coordinates with respect to the polar axis (that contains the line segment) OZ are $[r, \theta - \alpha]$. This situation is pictured in the below diagram.



Thus, the Cartesian coordinates of P with respect to the OZ-axis are $\big(r\cos(\theta - \alpha), r\sin(\theta - \alpha)\big)$. Because L consists of the points whose first coordinate in this system is d, this means that

$$r\cos(\theta - \alpha) = d. \qquad \square$$

Remark If $r < 0$ and $r\cos(\theta - \alpha) = d$, then the point $[r, \theta]$ is plotted at $[-r, \theta - \alpha + \pi]$. You may wonder if when we restrict to $r \geq 0$ we are missing some points on our line, but this is not the case. Indeed, we do not get any new points when $r < 0$ and this is made clear by the following:

$$(-r)\cos(\theta - \alpha + \pi) = r\cos(\theta - \alpha) = d.$$

5.2 Conics in Polar Coordinates

Theorem 5.2.1 *Consider a conic of eccentricity e, directrix L and focus F at a distance $d > 0$ away from L. Then, the equation of the conic in polar coordinates is*

$$r(1 - e\cos\theta) = ed.$$

Proof: Suppose first that $r \geq 0$ and let $P = [r, \theta]$ be a point in polar coordinates on the conic. Then, $PF = r$ and $\operatorname{dist}(P, L) = |d + r\cos\theta|$. Hence, the equation of the conic is

$$PF = e\operatorname{dist}(P, L) \qquad \Leftrightarrow \qquad r = e|d + r\cos\theta|.$$

We can re-write this as $r = \pm e(d + r\cos\theta) = r(\pm 1 - e\cos\theta) = ed$, were this holds because $r \geq 0$. Consider now the curve $r(1 - e\cos\theta) = ed$ and allow $r < 0$ as well as $r \geq 0$. If $[r, \theta]$ is a solution to this equation with $r < 0$, we can define $s := -r > 0$ and $\varphi := \theta + \pi$. Then, notice that

$$\begin{aligned} s(-1 - e\cos\varphi) &= -r\left(-1 - e\cos(\theta + \pi)\right) \\ &= r(1 - e\cos\theta) \\ &= ed. \end{aligned}$$

This explains the presence of the $\pm$ in the equation we wrote in the above paragraph. Hence, it follows that the equation of the conic is $r(1 - e\cos\theta) = ed$ where we now included $r < 0$ values. $\square$

5.3 Sketching Curves in Polar Coordinates

Definition A cardioid is the curve $r = a(1 + \cos\theta)$ for $0 \leq \theta < 2\pi$ and $a > 0$.

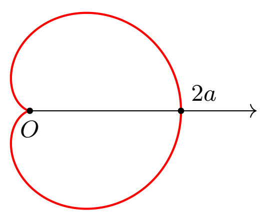


Figure 19: A sketch of a cardioid.

Definition A rose is either $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ for $0 \leq \theta < 2\pi$, $a \neq 0$ and $n \in \mathbb{Z}$.

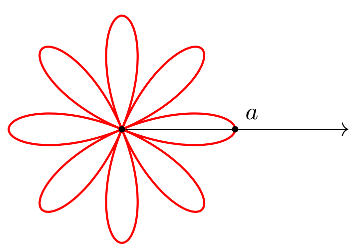


Figure 20: A sketch of the rose $r = a\sin(n\theta)$ for even n.

Note: If n is even, the rose curve has $2n$ petals and if n is odd, the curve has only n petals.

Definition The Archimedean spiral is the curve $r = \theta$ where $\theta \geq 0$.

The branches of the Archimedean spiral are a fixed distance of 2π apart, meaning that if $[r, \theta]$ lies on the curve, then so too does the point $[r + 2\pi, \theta]$.

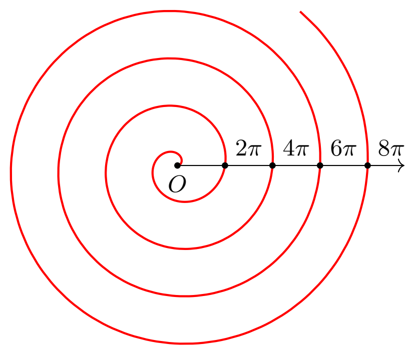


Figure 21: A sketch of the Archimedean spiral.

Definition The logarithmic spiral is the curve $r = e^{\theta}$ where $\theta \geq 0$.

If the logarithmic spiral is scaled down by $e^{2\pi}$, then the curve appears unchanged; this is a so-called *fractal property* of this curve. In other words, if $[r, \theta]$ lies on the curve, then so too does the point $[re^{-2\pi}, \theta]$. Furthermore, as $r \to 0$, we have $\theta \to -\infty$. But there is **no** value of θ for which $r = 0$; the curve nears the pole and winds tighter and tighter around it.

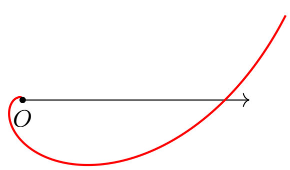


Figure 22: A sketch of the logarithmic spiral.

Method – Sketching Polar Curves: Suppose we want to sketch some polar equation.

(i) Try converting to Cartesian coordinates to see if it is easy to sketch there.
(ii) Consider $r \geq 0$ separately to $r < 0$ in order to simplify things.
(iii) Look for periodicity; this allows us to sketch part of the curve and then just rotate it to get a sketch of the entire curve.

6 Classification of Conics

6.1 Introduction

Reminder: A two-variable degree one curve has the form $ax+by+c=0$ for some $a,b,c \in \mathbb{R}$.

Recall that a degree one curve in two variables defines a straight line, except in the degenerate case where $a=b=0$, in which there are two possibilities:

- If $c=0$, then $0x+0y+c=0$ is satisfied by the entire plane, i.e. the curve is $\mathbb{R}^2$.
- If $c \neq 0$, then $0x+0y+c=0$ is satisfied by no point in the plane, i.e. the curve is $\emptyset$.

6.2 Classification via the Discriminant

Theorem 6.2.1 *Consider the curve defined by* $(*)$ *and suppose we apply a rotation of axes by* α *so that the equation of the conic transforms into*

$$aX^2+bXY+cY^2+dX+eY+f=0$$

for some $a,b,c,d,e,f \in \mathbb{R}$ *possibly different from* $A,B,C,D,E,F \in \mathbb{R}$ *respectively. Then, the* discriminant B^2-4AC *of the conic remains unchanged, that is*

$$b^2-4ac=B^2-4AC.$$

Proof: Recall the rotation of axes formulae from Corollary 4.3.2, which again apply here:

$$x=X\cos(\alpha)-Y\sin(\alpha) \qquad \text{and} \qquad y=X\sin(\alpha)+Y\cos(\alpha).$$

Note that we want to compute the discriminant, which requires only the coefficients of the second-order terms. Therefore, substituting the rotation of axes formulae into $(*)$ as we did in the proof of Lemma 4.4.1, the second-order terms we obtain are

$$A(X\cos\alpha-Y\sin\alpha)^2+B(X\cos\alpha-Y\sin\alpha)(X\sin\alpha+Y\cos\alpha)+C(X\sin\alpha+Y\cos\alpha)^2,$$

which we can expand and simplify so it has the form $aX^2+bXY+cY^2$, where

$$\begin{aligned} a &= A\cos^2(\alpha)+B\sin(\alpha)\cos(\alpha)+C\sin^2(\alpha),\\ b &= 2(C-A)\sin(\alpha)\cos(\alpha)+B(\cos^2\alpha-\sin^2\alpha),\\ c &= A\sin^2(\alpha)-B\sin(\alpha)\cos(\alpha)+C\cos^2(\alpha). \end{aligned}$$

It remains to simply compute the discriminant; after some calculations, we obtain this:

$$\begin{aligned} b^2-4ac &= (B^2-4AC)(\sin^4\alpha+2\sin^2\alpha\cos^2\alpha+\cos^4\alpha)\\ &= (B^2-4AC)(\sin^2\alpha+\cos^2\alpha)^2\\ &= B^2-4AC. \end{aligned}$$

□

Note: In the degenerate case where $B = 0$ in $(*)$, we have the following types of curve:

$$\begin{aligned} \textbf{Circle}&: \quad x^2 + y^2 = 1. \\ \textbf{Point}&: \quad x^2 + y^2 = 0. \\ \textbf{Empty Set}&: \quad x^2 + y^2 = -1. \\ \textbf{Parallel Lines}&: \quad x^2 = 1. \\ \textbf{One Line}&: \quad x^2 = 0. \\ \textbf{Empty Set}&: \quad x^2 = -1. \\ \textbf{Intersecting Lines}&: \quad x^2 - y^2 = 0. \end{aligned}$$

The first three are degenerate ellipses, the next three parabolas and the last a hyperbola.

Proposition 6.2.2 *Consider the curve defined by $(*)$ where $B = 0$, that is*

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

with the extra assumption that A and C are **not both** *zero. Then, we have the following:*

(i) *The curve is an ellipse if A and C have the same sign.*
(ii) *The curve is a hyperbola if A and C have opposite signs.*
(iii) *The curve is a parabola if one of A and C is zero.*

Proof: Assume first that $A, C \neq 0$. We can complete the square in both x and y to obtain

$$A(x + P)^2 + C(y + Q)^2 = R$$

for some $P, Q, R \in \mathbb{R}$. If we make the change of coordinates $X = x + P$ and $Y = y + P$, the equation of the curve above becomes $AX^2 + CY^2 = R$. We now have the following cases.

- Suppose A and C have the same sign. We assume (multiplying through by -1 if necessary) that $A, C > 0$. If $R < 0$, we obtain the empty set. If $R = 0$, we obtain a single point. If $R > 0$ **and** $A = C$, we obtain a circle. Otherwise, we obtain a standard ellipse.
- Suppose A and C have opposite signs. If $R = 0$, we obtain two intersecting lines. Otherwise, we obtain a standard hyperbola.

The final case is to assume that one of $A = 0$ and $C = 0$, with the other being non-zero.

- If $A = 0$ and $C \neq 0$, we can complete the square in y and divide through by C to obtain

  $$Y^2 = PX + Q$$

  for some $P, Q \in \mathbb{R}$. If $P = 0$ and $Q < 0$, we obtain the empty set. If $P = 0$ and $Q = 0$, we obtain one line. If $P = 0$ and $Q > 0$, we obtain two parallel lines. Otherwise, if $P \neq 0$, we obtain a standard hyperbola.
- If $A \neq 0$ and $C = 0$, we can argue identically to the above. □

Theorem 6.2.3 (Classification of Conics) *Any equation of the form* $(*)$ *with at least one of* A, B, C *being non-zero describes a (possibly degenerate) conic.*

(i) *If the discriminant* $B^2 - 4AC < 0$*, the conic is an ellipse.*

(ii) *If the discriminant* $B^2 - 4AC = 0$*, the conic is a parabola.*

(iii) *If the discriminant* $B^2 - 4AC > 0$*, the conic is a hyperbola.*

Proof: By Lemma 4.4.1, we can rotate the axes so that the mixed second-order term disappears, producing for us the equation $aX^2 + cY^2 + dX + eY + f = 0$. If $a = c = 0$, we know there can be no second-order terms since X and Y are linear in x and y, a contradiction. Therefore, a and c are **not both** zero. By Theorem 6.2.1, the discriminant is invariant under a rotation of axes, so the result follows from Proposition 6.2.2. □

7 Three-Dimensional Geometry

7.1 Introduction

In three dimensions, we require an additional axis compared to the usual situation of a plane. Indeed, here we introduce the z-axis. In this way, a point in three-dimensional space can be described in Cartesian coordinates as a triple of real numbers; this is analogous to how a point in the plane is a pair of real numbers.

Note: Three-dimensional space, as a set, is $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$.

An element of $\mathbb{R}^3$ can be thought of either as a point or a vector; we use both throughout.

7.2 Parametric Equation of a Line

Definition 7.2.1 A line in $\mathbb{R}^3$ is a subset consisting of (x, y, z) where, for some $t \in \mathbb{R}$,

$$(x, y, z) = (x_0, y_0, z_0) + t(u, v, w).$$

Here, (x_0, y_0, z_0) is a fixed point in $\mathbb{R}^3$ and the vector (u, v, w) is the direction of the line. We call t the parameter and the whole expression above the parametric equation of a line.

Notation We sometimes use the following notation for the parametric equation of a line:

$$X = X_0 + tU,$$

where $X = (x, y, z)$, $X_0 = (x_0, y_0, z_0)$ is the fixed point and $U = (u, v, w)$ is the direction vector.

Lemma 7.2.2 *Let $X_0 = (x_0, y_0, z_0)$ and $X_1 = (x_1, y_1, z_1)$ be distinct points in $\mathbb{R}^3$. Then, the line through X_0 and X_1 is given by*

$$X = X_0 + t(X_1 - X_0).$$

Proof: By Definition 7.2.1, the line has the form $X = X_0 + sU$ where $s \in \mathbb{R}$ is the parameter and U is the direction vector. Since X_1 lies on the line, there is a value $s = s'$ such that

$$X_1 = X_0 + s'U.$$

Since X_0 and X_1 are distinct, we know that $s' \neq 0$. This allows us to rearrange this to get

$$U = \frac{1}{s'}(X_1 - X_0).$$

Hence, substituting this and setting $t := s/s'$, we get $X = X_0 + t(X_1 - X_0)$ as expected. □

7.3 Implicit Equation of a Plane

Definition 7.3.1 A plane in $\mathbb{R}^3$ is a set of points (x, y, z) that satisfy the equation

$$ax + by + cz = d$$

for some $a, b, c, d \in \mathbb{R}$ where a, b, c are **not all** zero.

Remark Like with some of the degenerate cases we have seen before, note that if $a = b = c = 0$, the above equation describes the whole space $\mathbb{R}^3$ when $d = 0$, and the empty set $\emptyset$ when $d \neq 0$.

Method – Equation of a Plane: Suppose we have three specified points and we wish to determine the equation of the plane that contains all of these points. Then, we simply substitute each of the points (x, y, z) into the equation $ax + by + cz = d$ and then solve the resulting simultaneous equations for $a, b, c, d \in \mathbb{R}$.

Note: For all $k \neq 0$, $ax + by + cz = d$ and $kax + kby + kcz = kd$ define the same plane.

7.4 Distance, Angles and the Dot Product

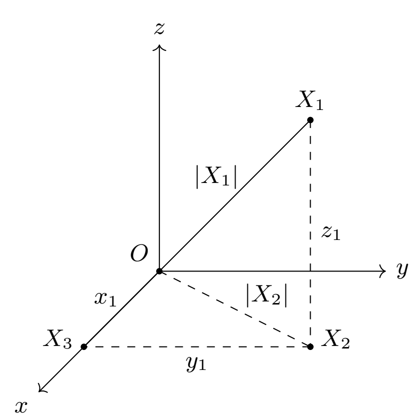
Definition 7.4.1 The distance between two points X_0 and X_1 is the length of the line segment between them, which we denote by $\text{dist}(X_0, X_1)$ or $d(X_0, X_1)$ or $|X_1 - X_0|$.

Notation If $X_0 = O = (0, 0, 0)$ is the origin, the distance between X_0 and X_1 is written $|X_1|$.

Theorem 7.4.2 *Let $X_0 = (x_0, y_0, z_0)$ and $X_1 = (x_1, y_1, z_1)$ be points in $\mathbb{R}^3$. Then,*

$$|X_1 - X_0| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

Proof: Let $X_0 = (0, 0, 0)$ for the moment; the general case is similar. In this situation, we must show the distance to the origin is $|X_1| = \sqrt{x_1^2 + y_1^2 + z_1^2}$. Now, consider the following diagram.



Here, we have defined some new points $X_2 := (x_1, y_1, 0)$ and $X_3 := (x_1, 0, 0)$ (the sketch applies when $x_1, y_1, z_1 > 0$ but similar drawings can be made for the other situations). Because OX_1X_2 and OX_2X_3 are each right-angled triangles, Pythagoras' Theorem implies

$$|X_1|^2 = |X_2|^2 + z_1^2 \qquad \text{and} \qquad |X_2|^2 = x_1^2 + y_1^2.$$

Substituting the second into the first and taking the square root gives the result. □

Definition 7.4.3 The dot (or scalar) product of $X_0 = (x_0, y_0, z_0)$ and $X_1 = (x_1, y_1, z_1)$ is

$$X_0 \cdot X_1 := x_0x_1 + y_0y_1 + z_0z_1.$$

Lemma 7.4.5 *These properties of the dot product hold for any vectors X_0, X_1, X_2 in $\mathbb{R}^3$:*

(i) $X_0 \cdot X_0 = |X_0|^2$.

(ii) $(-X_0) \cdot X_1 = -(X_0 \cdot X_1) = X_0 \cdot (-X_1)$. (**Homogeneity**)

(iii) $X_0 \cdot X_1 = X_1 \cdot X_0$. (**Commutativity**)

(iv) $(X_0 + X_1) \cdot X_2 = (X_0 \cdot X_2) + (X_1 \cdot X_2)$. (**Distributivity**)

Sketch of Proof: (i) Recall that the magnitude $|X_0| = \sqrt{x_0^2 + y_0^2 + z_0^2}$ by Theorem 7.4.2. If we square this, it gives us precisely $x_0^2 + y_0^2 + z_0^2$, which is just Definition 7.4.3 with $X_2 = X_1$.

(ii) Just apply the definition with $X_0 = (x_0, y_0, z_0)$ and $X_1 = (x_1, y_1, z_1)$.

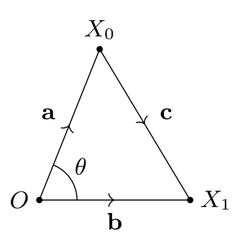
(iii) This follows from the fact that multiplication of two real numbers is commutative.

(iv) Just apply the definition with $X_0 = (x_0, y_0, z_0)$, $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$. □

Theorem 7.4.6 *Let X_0 and X_1 be points in $\mathbb{R}^3$ distinct from the origin and θ be the angle between the line segments OX_0 and OX_1, meaning that $0 \leq \theta < \pi$. Then,*

$$X_0 \cdot X_1 = |X_0||X_1| \cos(\theta).$$

Proof: Let $\mathbf{a}$ be the vector OX_0 and $\mathbf{b}$ be the vector OX_1. This means the vector X_0X_1 is defined as $\mathbf{c} = \mathbf{b} - \mathbf{a}$. If $\mathbf{a}$ and $\mathbf{b}$ are parallel, meaning $\mathbf{a} = k\mathbf{b}$ for some non-zero $k \in \mathbb{R}$, the result is immediate since $\theta = 0$. Otherwise, we turn to the set-up pictured below.



If we apply the Cosine Rule (Theorem 1.15.1) to triangle OX_0X_1, we see that

$$\begin{aligned}|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta) &= |\mathbf{c}|^2 \\ &= |\mathbf{b} - \mathbf{a}|^2 \\ &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \cdot (\mathbf{b} - \mathbf{a}) - \mathbf{a} \cdot (\mathbf{b} - \mathbf{a}) \\ &= (\mathbf{b} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{a}) \\ &= |\mathbf{a}|^2 - 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2,\end{aligned}$$

where we use the properties in Lemma 7.4.5. Consequently, we can rearrange this to get

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta).$$

All that remains is to note that we can alternatively denote $\mathbf{a} = X_0$ and $\mathbf{b} = X_1$. □

Note: If instead X_0 and X_1 are **vectors** (not just points), then the angle between them is still defined as the angle between OX_0 and OX_1 where we regard X_0 and X_1 here as points. This is a technicality; we often not distinguish between points and vectors in $\mathbb{R}^3$.

Definition 7.4.7 Two vectors X_0 and X_1 are orthogonal if their dot product $X_0 \cdot X_1 = 0$.

Remark 7.4.8 By Theorem 7.4.6, two non-zero vectors are orthogonal if and only if the angle between them is $\theta = \pi/2$. Be careful when calculating dot products: if we have $X_0 \cdot X_1 = k$, then we **cannot** write something like $X_0 = k/X_1$. This is because it doesn't make sense to divide a number by a vector; the dot product is not multiplication, but a special type of operation.

Method – Angle Between Vectors: Let us find the angle between the vectors U and V.

(i) Compute their dot product $U \cdot V$ and their magnitudes $|U|$ and $|V|$.
(ii) Substitute these into Theorem 7.4.6.
(iii) Rearrange and apply $\cos^{-1}$ to obtain the angle θ between U and V.

7.5 Angle Between Two Lines

Definition 7.5.1 Let $X = X_0 + tU_0$ and $X = X_1 + tU_1$ be parametric equations of two lines in $\mathbb{R}^3$. The angle between the lines is the angle between direction vectors U_0 and U_1.

Method – Angle Between Lines: To find the angle between two lines, just apply the above method for the direction vectors; this is immediate from Definition 7.5.1.

Remark There are two angles between a pair of lines, namely θ and $\pi - \theta$. However, we will always take θ as the angle given in the parametric form as i Definition 7.5.1. For $-1 \leq a \leq 1$, we also adopt the convention that $\cos^{-1}(a)$ is the unique number $0 \leq \theta \leq \pi$ where $\cos(\theta) = a$.

7.6 The Cross Product

Definition 7.6.1 The cross (or vector) product of $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ is

$$A \times B := (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

Remark 7.6.2 We can 'build' $\mathbb{R}^3$ from the vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$ in the sense that any $(x, y, z) \in \mathbb{R}^3$ can be written as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. This is an example of a set of so-called *basis vectors*. This particular set is called the *standard basis* of $\mathbb{R}^3$. Using these, we can re-write the cross product as a matrix determinant (if you haven't seen this yet, don't worry):

$$A \times B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Note: The so-called "right-hand rule" tells us in which way the cross product will point. Indeed, point your thumb upwards and extend your index finger forwards and middle finder to the left. If those fingers represent A and B, your thumb represents $A \times B$.

Remark 7.6.4 Notice that the dot product is also called the *scalar* product (because it produces a scalar) and, similarly, the cross product is called the *vector* product (since it produces a vector).

Theorem 7.6.5 *These properties of the cross product hold for any vectors A, B in $\mathbb{R}^3$:*

(i) $A \cdot (A \times B) = 0$.
(ii) $B \cdot (A \times B) = 0$.
(iii) $A \times B = -B \times A$.
(iv) $|A \times B|^2 = |A|^2|B|^2 - (A \cdot B)^2$.

Sketch of Proof: Simply use the formula in Definition 7.6.1 (and Definition 7.4.3). □

Remark 7.6.6 In words, Theorem 7.6.5(i) and (ii) tell use that $A \times B$ is orthogonal to both A and B (as is suggested by the "right-hand rule"). More over, we immediately see from Theorem 7.6.5(iii) that $A \times A = (0, 0, 0)$. This generalises to a slightly stronger statement: the cross product of two **parallel** vectors is zero, meaning $A \times (\lambda A) = (0, 0, 0)$ for any $\lambda \in \mathbb{R}$.

Proposition 7.6.7 *Let A and B be vectors in $\mathbb{R}^3$ and θ be the angle between them. Then,*

$$|A \times B| = |A||B| \sin(\theta).$$

Proof: From Theorem 7.6.5(iv) combined with Theorem 7.4.6, we see that

$$|A \times B|^2 = |A|^2|B|^2 - (A \cdot B)^2 = |A|^2|B|^2 - |A|^2|B|^2 \cos^2(\theta) = |A|^2|B|^2 \sin^2(\theta).$$

Finally, we can take the square root since $\sin(\theta) \geq 0$ since we assume $0 \leq \theta \leq \pi$. □

7.7 A Better Way to Find the Equation of a Plane

Definition 7.7.1 A normal vector to the a is a non-zero vector N which is orthogonal to every vector in the plane, meaning $N \cdot (P_1 - P_2) = 0$ for any points P_1 and P_2 in the plane.

Lemma 7.7.2 *If a plane is given by $ax + by + cz = d$ with a, b, c* **not all** *zero, then (a, b, c) is a normal vector to the plane.*

Proof: Let $N = (a, b, c)$, meaning the equation of the plane is $N \cdot P = d$ where $P = (x, y, z)$ is a general point. If P_1 and P_2 are two points in the plane, we know that $N \cdot P_1 = d$ and $N \cdot P_2 = d$. Subtracting the second from the first tells us $N \cdot (P_1 - P_2) = 0$. Hence, N is a normal vector. □

Theorem 7.7.3 *Let P_1, P_2 and P_3 be distinct points in a plane that are not collinear. Then, any normal vector N to the plane is a non-zero multiple of the vector*

$$(P_2 - P_1) \times (P_3 - P_1).$$

Proof: Since the points are distinct, we know that $P_2 - P_1$ and $P_3 - P_1$ are non-zero. Because the points are not collinear, the angle between the vectors is not 0 nor π. If we now define $M := (P_2 - P_1) \times (P_3 - P_1)$, we know from Theorem 7.6.5 that M is orthogonal to both $P_2 - P_1$ and $P_3 - P_1$. But by Definition 7.7.1, we have $N \cdot (P_2 - P_1) = 0$ and $N \cdot P_3 - P_1 = 0$. Thus, N will also be proportional to M. □

Method – Equation of a Plane: Suppose we have three specified points P_1, P_2 and P_3 and we wish to determine the equation of the plane that contains all of these points.

(i) Subtract one of the vectors from each of the others, e.g. $P_2 - P_1$ and $P_3 - P_1$.
(ii) Use the formula in Definition 7.6.1 to compute $(P_2 - P_1) \times (P_3 - P_1) = (a, b, c)$.
(iii) Substitute one of the points into $ax + by + cz = d$ to compute d.
(iv) If necessary, divide through by $\text{hcf}(a, b, c, d)$ to get it in the simplest form.

Note: To find the angle θ between a line and plane that intersect, simply compute the angle α between the line's direction vector and the plane's normal vector as discussed earlier and then subtract it from $\pi/2$. This can be pictured in Figure 23 below.

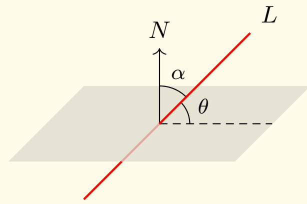


Figure 23: Finding the angle between a line L and a plane.

8 Polyhedra

8.1 Introduction

Definition 8.1.1 A polygon is a cyclic path in the plane consisting of line segments with no self-intersections and where no two successive line segments are parallel. Such a line segment is called an edge and the place where two successive line segments meet is called a vertex. If there are n edges, we call the shape an n-gon.

Definition A polygon is called convex if the line segment between any two points on its edges lies within the interior of the polygon.

(a) A convex polygon. (b) A non-convex polygon.

Figure 24: Convexity of some n-gons.

Theorem 8.1.2 *The interior angles of a convex n-gon sum to $(n-2)\pi$.*

Proof: Let X be a point in the interior of the n-gon and draw line segments from X to each of the vertices of the polygon; this divides the polygon into n triangles. We know from Proposition 1.10.5 that the sum of the angles in each triangle is π. But this is also equal to the sum S of the interior angles and the sum of the angles around X. In other words,

$$S + 2\pi = n\pi \qquad \Rightarrow \qquad S = (n-2)\pi. \qquad \square$$

Note: It turns out Theorem 8.1.2 holds for non-convex polygons (but we don't need this).

Definition 8.1.3 A polygon is called regular if both of the following hold true:
(i) All sides are the same length.
(ii) All interior angles are equal.

Theorem 8.1.4 *A single interior angle of a regular n-gon is equal to $\frac{n-2}{n}\pi$.*

Proof: Let α be an interior angle. By Theorem 8.1.2, we have $n\alpha = (n-2)\pi$ and so $\alpha = \frac{n-2}{n}\pi$. $\square$

8.2 Platonic Solids

Definition 8.2.1 A polyhedron is a closed figure in three-dimensional space with a boundary consisting of a finite number of (non-parallel) polygons. We call each of these polygons a face, and still use "vertex" and "edge" in the way introduced in Definition 8.1.1. The intersection of any two faces must either be a common vertex, a common edge or empty.

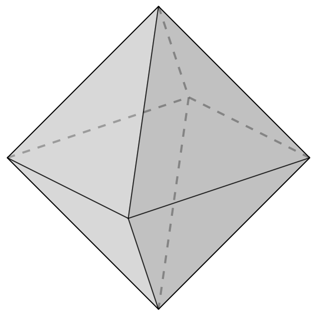


Figure 25: An example of a polyhedron built from eight triangles (3-gons).

Definition 8.2.2 A polyhedron is called convex if the line segment between any two points on its boundary lies within the inside of the polyhedron.

Definition 8.2.3 A polyhedron is called regular if both of the following hold true:

(i) All faces are pairwise congruent regular polygons.

(ii) The number of faces meeting each vertex is the same.

We can classify all polyhedrons that are both convex and regular just be counting certain things.

Theorem 8.2.4 *There are only five* Platonic solids *(polyhedra that are convex* **and** *regular), which are described in the below table where v is the total number of vertices, e is the total number of edges, f is the total number of faces, p is the number of edges on each face and q is the number of faces meeting each vertex:*

	v	e	f	p	q
Tetrahedron	4	6	4	3	3
Cube	8	12	6	4	3
Octahedron	6	12	8	3	4
Dodecahedron	20	30	12	5	3
Icosahedron	12	30	20	3	5

Proof: The idea here is to find a numerical condition that all Platonic solids must satisfy; we then go through all possibilities satisfying this condition. Indeed, suppose that the faces are p-gons and q of them meet at each vertex. Because the polyhedron is convex, the angles at a vertex sum to less than 2π. Note each of the angles is equal to $\frac{p-2}{p}\pi$ by Theorem 8.1.4. Hence,

$$\begin{aligned} & \left(\frac{p-2}{p}\pi\right) q < 2\pi \\ \Leftrightarrow \quad & (p-2)q < 2p \\ \Leftrightarrow \quad & pq - 2q - 2p < 0 \\ \Rightarrow \quad & pq - 2q - 2p + 4 < 4 \\ \Leftrightarrow \quad & (p-2)(q-2) < 4. \end{aligned}$$

This tells us we must have that $(p-2)(q-2) \in \{1, 2, 3\}$, but $p, q \geq 3$ are positive integers so there are not too many possibilities that need to be considered.

(i) If $(p-2)(q-2) = 1$, this means that $p-2 = q-2 = 1$ and thus $p = q = 3$. In other words, we have equilateral triangles with three at a vertex. This forms a **tetrahedron**.

(ii) If $(p-2)(q-2) = 2$, there are two possibilities:

- $p - 2 = 2$ and $q - 2 = 1$, meaning $p = 4$ and $q = 3$. This forms a **cube**.
- $p - 2 = 1$ and $q - 2 = 2$, meaning $p = 3$ and $q = 4$. This forms an **octahedron**.

(iii) If $(p-2)(q-2) = 3$, there are two possibilities:

- $p - 2 = 3$ and $q - 2 = 1$, meaning $p = 5$ and $q = 3$. This forms a **dodecahedron**.
- $p - 2 = 1$ and $q - 2 = 3$, meaning $p = 3$ and $q = 5$. This forms an **icosahedron**. □

8.3 Euler's Formula

Theorem 8.3.1 (Euler's Formula) *If we have a polyhedron that deforms into a sphere, then*

$$v - e + f = 2,$$

where v is the number of vertices, e is the number of edges and f is the number of faces.

Proof: (**Non-examinable**) Suppose we construct a polyhedron one face at the time. The key idea is that we can do this in such a way that whenever we add a new face, it is glued to the old faces along a chain of consecutive edges. Suppose v_k and e_k are the respective numbers of vertices and edges after adding the k^{th} face. We will consider $e_k - v_k$.

If the first face is an n-gon, it has n vertices and n edges, then $e_1 - v_1 = 0$. Whenever we add a face (other than the last one), we join it with the precious faces along a chain of consecutive edges; this adds a new chain of edges. Hence, the number of edges added is one more than the number of vertices. This means for $2 \leq k \leq f - 1$, we have

$$e_k - v_k = e_{k-1} - v_{k-1} + 1.$$

It follows that $e_{f-1} - v_{f-1} = f - 2$. But for the final face, we add **no** new edges or vertices, meaning $e_f - v_f = f - 2$. But since $v_f = v$ and $e_f = e$, this rearranges to give the result. $\square$

Note: The part in Euler's Formula where we impose that our polyhedron must deform into a sphere is necessary to exclude some strange examples with 'holes'. One such case of a polyhedron which does **not** deform into a sphere is that of a *toroid* as drawn below.