MATH1026 Sets, Sequences and Series

Cheatsheet

2023/24

This document collects together the important definitions and results presented throughout the lecture notes. The numbering used throughout will be consistent with that in the lecture notes.

Contents

1 The Set of Real Numbers

Reminder: We will use the following notation throughout the module:

$$\mathbb{N} = \{1, 2, 3, 4, ...\} \qquad \text{and} \qquad \mathbb{N}_0 = \{0, 1, 2, 3, 4, ...\}.$$

1.1 Axiomatic Characterisation of the Real Numbers

Definition 1.1.1 A field is a set A with two binary operations $+ : A \times A \to A$ (addition) and $\cdot : A \times A \to A$ (multiplication) such that the following axioms are all satisfied:

(A1) For all $a, b, c \in A$, $(a + b) + c = a + (b + c)$. (**Associativity**)
(A2) For all $a, b, c \in A$, $a + b = b + a$. (**Commutativity**)
(A3) There exists an element $0 \in A$ with $0 + a = a$ for all $a \in A$. (**Identity**)
(A4) For each $a \in A$, there exists $y \in A$ with $a + y = 0$. (**Inverses**)
(M1) For all $a, b, c \in A$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (**Associativity**)
(M2) For all $a, b, c \in A$, $a \cdot b = b \cdot a$ (**Commutativity**)
(M3) There exists an element $1 \in A$ with $a \cdot 1 = 1$ for all $a \in A \setminus \{0\}$. (**Identity**)
(M4) For each $a \in A \setminus \{0\}$, there exists $x \in A$ with $a \cdot x = 1$. (**Inverses**)
(D1) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. (**Distributivity**)

Notation We relabel the additive and multiplicative inverses to reference the element they are relate to: the additive and multiplicative inverses of a are denoted $-a$ and a^{-1}, respectively.

Theorem 1.1.2 *Let A be a field. Then, we have the following:*

(i) *The additive identity $0 \in A$ that exists by Axiom (A3) is unique.*
(ii) *The multiplicative identity $1 \in A$ that exists by Axiom (M3) is unique.*
(iii) *The additive inverse $-a$ of $a \in A$ that exists by Axiom (A4) is unique.*
(iv) *The multiplicative inverse a^{-1} of $a \in A$ that exists by Axiom (M4) is unique.*

Proof: (i) Let $x \in A$ such that $a + x = a$ for all $a \in A$. In particular, this equality works for $a = 0$, that is $0 + x = 0$. On the other hand, adding 0 to any element doesn't change it by Axiom (A3), so we know that $x + 0 = x$. But commutativity in Axiom (A2) tells us that $x + 0 = 0 + x$. Hence, we can stitch these two equations together to obtain $x = x + 0 = 0 + x = 0$.

(ii) Omitted; this is analogous to the proof of part (i).

(iii) Let $a \in A$ be arbitrary and suppose that $x, y \in A$ both satisfy Axiom (A4), that is $a + x = 0$ and $a + y = 0$. We can stitch these two equations together using Axioms (A3), (A1) and (A2) to produce $x = x + 0 = x + (a + y) = (x + a) + y = 0 + y = y$.

(iv) Omitted; this is analogous to the proof of part (iii). □

Note: Multiplication notation is often suppressed by writing $ab := a \cdot b$ without the dot.

Theorem 1.1.3 *Let A be a field. Then, we have the following for all $a, b, c \in A$:*

(i) $a + c = b + c$ *implies* $a = b$.
(ii) $a \cdot 0 = 0$.
(iii) $(-a)b = -(ab)$.
(iv) $(-a)(-b) = ab$.
(v) $ac = bc$ *with* $c \neq 0$ *implies* $a = b$.
(vi) $ab = 0$ *implies* $a = 0$ *or* $b = 0$.

Proof: (i) Starting from the equality in question, we have

$$\begin{aligned} a + c = b + c \quad &\Rightarrow \quad a + (c + (-c)) = b + (c + (-c)) \\ &\Leftrightarrow \quad a + 0 = b + 0, && \text{by Axiom (A4)}, \\ &\Leftrightarrow \quad a = b. \end{aligned}$$

(ii) We just simply apply what we proved in part (i) to the following:

$$\begin{aligned} 0 + (a \cdot 0) &= a \cdot 0, && \text{by Axiom (A3)}, \\ &= a \cdot (0 + 0), && \text{by Axiom (A3) again}, \\ &= (a \cdot 0) + (a \cdot 0), && \text{by Axiom (D1)}. \end{aligned}$$

(iii) Axiom (A4) and the uniqueness of inverses from Theorem 1.1.2(iii) imply the result from

$$\begin{aligned} (-a)b + ab &= ((-a) + a)b, && \text{by Axiom (D1)}, \\ &= 0b, && \text{by Axiom (A3)}, \\ &= 0, && \text{by Axiom (A2) and part (ii)}. \end{aligned}$$

(iv) Let's work from the left-hand side to the right in the following way:

$$\begin{aligned} (-a)(-b) &= (-a)(-b) + 0, && \text{by Axiom (A3)}, \\ &= (-a)(-b) + 0b, && \text{by part (ii)}, \\ &= (-a)(-b) + ((-a) + a)b, && \text{by Axiom (A4)}, \\ &= (-a)(-b) + (-a)b + ab, && \text{by Axiom (D1)}, \\ &= (-a)((-b) + b) + ab, && \text{by Axiom (D1)}, \\ &= ((-a) \cdot 0) + ab, && \text{by Axiom (A4)}, \\ &= 0 + ab, && \text{by part (ii)}, \\ &= ab, && \text{by Axiom (A3)}. \end{aligned}$$

(v) All one must do is multiply on the right by c^{-1} and cancel.

(vi) Let $ab = 0$. If $b = 0$, we are done. On the other hand, if $b \neq 0$, we multiply by b^{-1} on the right. In this way, we see that $abb^{-1} = 0b^{-1} = 0$, but part (ii) and Axiom (M3) gives us the alternate expression $abb^{-1} = a \cdot 1 = a$. Equating allows us to immediately conclude $a = 0$. □

Reminder: A binary relation on a set X is a subset $R \subseteq X \times X$ of the Cartesian product of the set with itself. For $x, y \in X$, we say that x is related to y if the pair $(x, y) \in R$.

Definition 1.1.4 An ordered field is a field A together with a binary relation $\leq$ on A that satisfies the following for all $a, b, c \in A$:
(O1) $a \leq b$ or $b \leq a$.
(O2) $a \leq b$ and $b \leq c$ implies $a \leq c$.
(O3) $a \leq b$ and $b \leq a$ implies $a = b$.
(O4) $a \leq b$ implies $a + c \leq b + c$.
(O5) $a \leq b$ and $c \geq 0$ implies $ac \leq bc$.

Note: We will use the notation $x < y$ to denote the fact that $x \leq y$ and $x \neq y$.

Theorem 1.1.5 *For any $a, b, c \in A$ elements of an ordered field, we have the following:*
(i) $a \leq b$ *implies* $-b \leq -a$.
(ii) $a \leq b$ *and* $c \leq 0$ *implies* $ac \geq bc$.
(iii) $a \leq 0$ *and* $b \leq 0$ *implies* $ab \geq 0$.
(iv) $a^2 \geq 0$.
(v) $1 > 0$.
(vi) $a > 0$ *implies* $a^{-1} > 0$.
(vii) $0 < a < b$ *implies* $0 < b^{-1} < a^{-1}$.

Sketch of Proof: (i) Add $(-a) + (-b)$ to both sides and use Axiom (O4).

(ii) Use the fact $-c \geq 0$ from part (i) and combine Axiom (O5) with part (i) again.

(iii) Use Axiom (O5) to obtain $-ba \leq 0$ and then use part (i).

(iv) The case where $a \geq 0$ follows from Axiom (O5), and $a \leq 0$ follows from part (iii).

(v) Assume to the contrary that $1 \not> 0$, that is $1 \leq 0$. Then, part (iii) implies that $0 \leq 1 \cdot 1$ (substituting $a = b = 1$), which violates Axiom (O3) because $0 \neq 1$.

(vi) Assume to the contrary that $a > 0$ but $a^{-1} \leq 0$. Then, part (iii) implies that $1 = aa^{-1} \leq 0$ which contradicts part (v).

(vii) From part (vi), we know $a^{-1} > 0$. Multiplying the inequality by this and using Axiom (O5) gives us $0a^{-1} < aa^{-1} < ba^{-1}$, which is equivalent to $0 < 1 < ba^{-1}$. But because $b^{-1} > 0$ again by part (vi), we can multiply this neq inequality by this to conclude that $0b^{-1} < 1b^{-1} < ba^{-1}b^{-1}$ which is $0 < b^{-1} < a^{-1}$ by using commutativity from Axiom (M3). $\square$

Although you should take care to remember these results, they should be very familiar; this is the sort of thing we have been working with since high school (or even before) except now we are interested in **any** ordered field, not just $\mathbb{R}$. We haven't proved that $\mathbb{R}$ is an ordered field though!

Definition 1.1.7 Let A be an ordered field and $S \subseteq A$ a subset.

- We say S is bounded above if there exists $M \in A$ such that $s \leq M$ for all $s \in S$.
- We say S is bounded below if there exists $L \in A$ such that $s \geq L$ for all $s \in S$.
- We say S is bounded if it is both bounded above and below.

In these cases, we call M an upper bound and L a lower bound for S.

Remark 1.1.8 For a subset $S \subseteq A$ of an ordered field, we can define the following new subset:

$$-S := \{-s : s \in S\} \subseteq A.$$

Then, S is bounded above if and only if $-S$ is bounded below (and vice versa). As one may expect, an upper/lower bound on $-S$ will turn out to be a lower/upper bound on S, respectively.

Definition 1.1.9 Let A be an ordered field and $S \subseteq A$ a subset. The supremum is the least upper bound on the set S, that is an upper bound M with the property that for all $x \in A$ which is also an upper bound on S, we have $M \leq x$. We denote this by $M = \sup(S)$.

Remark In practice, we often use the *contrapositive* of the second condition in Definition 1.1.9, namely to show some M is the *least* upper bound on $S \subseteq A$, it suffices to prove the following: "for any $x \in A$ such that $x < M$, then x is **not** an upper bound on S".

Proposition 1.1.11 *If a subset $S \subseteq A$ of an ordered field* **contains** *an upper bound $M \in S$, then said upper bound $M = \sup(S)$.*

Proof: By assumption, M is an upper bound so we need only show it is the *least* upper bound. To that end, it suffices to show that any $x \in A$ with $x < M$ is **not** an upper bound on S. But this is clear because $x < M$ is equivalent to $x \ngeq M$, so x is **not** an upper bound on S because there exists an element of this subset (namely M) which is larger than x. □

Note: If S is finite, a least upper bound **always** exists, namely the largest element of S.

Definition 1.1.13 Let A be an ordered field and $S \subseteq A$ a subset. The infimum is the greatest lower bound on the set S, that is a lower bound L with the property that for all $x \in A$ which is also a lower bound on S, we have $L \geq x$. We denote this by $L = \inf(S)$.

Remark Again, we often use the *contrapositive* of the second condition in Definition 1.1.13, namely to show some L is the *greatest* lower bound on $S \subseteq A$, it suffices to prove the following: "for any $x \in A$ such that $x > M$, then x is **not** a lower bound on S".

Definition 1.1.14 An ordered field A is called complete if **every** non-empty subset that is bounded above has a supremum in A.

Remark 1.1.15 We can actually rephrase completeness in terms of lower bounds and infima: "an ordered field is complete if every non-empty subset that is bounded *below* has an *infimum* in A". It turns out that this is completely equivalent to Definition 1.1.14.

Axiom (Axiom of Completeness) $\mathbb{R}$ *with the usual operations is a complete ordered field.*

The Axiom of Completeness implies the infimum-version of completeness from Remark 1.1.15.

Proposition 1.1.16 *Every non-empty subset of* $\mathbb{R}$ *bounded below has an infimum in* $\mathbb{R}$.

Proof: Let $S \subseteq \mathbb{R}$ be non-empty and bounded below by $L \in \mathbb{R}$. Then, Remark 1.1.8 tells us that $-S = \{-s : s \in S\} \subseteq \mathbb{R}$ is bounded **above** by $-L$. Indeed, we have $L \leq s$ all $s \in S$, and any $y \in -S$ can be written as $y = -s$ for some $s \in S$. Hence, $y = -s \leq -L$ by Theorem 1.1.5(i). By the Axiom of Completeness, we know that $-S$ has a *least* upper bound (a supremum), say $M = \sup(-S)$. We claim that $-M$ is the infimum of S. Well, because M is an upper bound on $-S$, we know that $-s \leq M$ for all $s \in S$. But this is equivalent to $s \geq -M$ for all $s \in S$, so we know that $-M$ is a lower bound on S. It remains to show that any number greater than it is **not** a lower bound. Indeed, let $x \in \mathbb{R}$ with $x > -M$. This means that $-x < M$ but because M is the greatest upper bound on $-S$, $-x$ is **not** an upper bound. Hence, there exists $s \in S$ such that $-x < -s \in -S$; this is equivalent to $x > s \in S$ so x is **not** a lower bound on S. ☐

Reminder: The set of rational numbers is the set of quotients of integers, that is

$$\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ with } b \neq 0\right\}.$$

Proposition 1.1.17 $\mathbb{Q}$ *with the usual operations is a* **non**-*complete field.*

Sketch of Proof: Showing it is a field is not too challenging. As for the lack of the completeness property, it suffices to show that the subset $S = \{x \in \mathbb{Q} : x^2 < 2\}$ does **not** have a supremum in the rationals. ☐

1.2 Properties of the Real Field

Theorem 1.2.2 (Archimedean Property of $\mathbb{R}$) *For all* $x \in \mathbb{R}$, *there exists* $N \in \mathbb{N}$ *with* $N > x$.

Proof: Assume to the contrary this is not the case, i.e. there exists $x \in \mathbb{R}$ such that, for all $N \in \mathbb{N}$, we have $N \geq x$. In other words, this says the set $\mathbb{N}$ is bounded above by x. By the Axiom of Completeness, $\mathbb{N}$ has a least upper bound $y \in \mathbb{R}$. By definition, $y - 1$ is **not** an upper bound on $\mathbb{N}$, which means there is an element of the naturals larger than it; there exists $M \in \mathbb{N}$ with $M \geq y - 1$. This is equivalent to $y \leq M + 1$, but $M + 1 \in \mathbb{N}$, so y is **not** an upper bound on $\mathbb{N}$, a contradiction to it being the least upper bound. ☐

Corollary 1.2.3 *For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $0 < \frac{1}{N} < \varepsilon$.*

Proof: By the Archimedean Property of $\mathbb{R}$, we can choose $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$. □

Corollary 1.2.4 *For all $\delta > 0$ and $x > 0$, there exists $N \in \mathbb{N}$ with $N\delta > x$.*

Proof: By the Archimedean Property of $\mathbb{R}$, we can choose $N \in \mathbb{N}$ such that $\frac{x}{\delta} < N$. □

Proposition 1.2.1 *There exists $y \in \mathbb{R}$ such that $y^2 = 2$.*

Proof: Let $S = \{x \in \mathbb{R} : x^2 < 2\}$, $y := \sup(S)$ and assume to the contrary that $y^2 \neq 2$.

- Let $y^2 < 2$. We aim to find an element of S which is larger than y as to contradict the fact y is an upper bound on S. One way to do this is to find $N \in \mathbb{N}$ with $(y + \frac{1}{N})^2 < 2$. Well,

$$\left(y + \frac{1}{N}\right)^2 = y^2 + \frac{2y}{N} + \frac{1}{N^2} \leq y^2 + \frac{2y}{N} + \frac{1}{N} \leq y^2 + \frac{5}{N}$$

  since $y < 2$ (this is *an* upper bound on S). Therefore, to ensure that $(y + \frac{1}{N})^2 < 2$, it suffices to pick $N \in \mathbb{N}$ such that $y^2 + \frac{5}{N} < 2$. Rearranging this inequality tells us that

$$N > \frac{5}{2 - y^2}$$

  which exists by the Archimedean Property of $\mathbb{R}$. Hence, we have such an element of S and y is not an upper bound; this is a contradiction.

- Let $y^2 > 2$. We aim to find an upper bound on S larger than y as to contradict the fact y is the *least* upper bound on S. For this, we can find $N \in \mathbb{N}$ with $(y - \frac{1}{N})^2 > 2$. Indeed,

$$\left(y - \frac{1}{n}\right)^2 = y^2 - \frac{2y}{N} + \frac{1}{N^2} \geq y^2 - \frac{2y}{N} \geq y^2 - \frac{4}{N}$$

  again since $y < 2$. In order to ensure that $(y - \frac{1}{N})^2 > 2$, we can simply find $N \in \mathbb{N}$ such that $y^2 - \frac{4}{N} > 2$. Rearranging this inequality tells us that

$$N > \frac{4}{y^2 - 2}$$

  which also exists by the Archimedean Property of $\mathbb{R}$. Hence, $y - \frac{1}{N}$ is an upper bound on S but clearly $y - \frac{1}{N} < y$; this is a contradiction. □

Lemma 1.2.6 (Well-Ordering of $\mathbb{N}$) *Every non-empty subset of $\mathbb{N}$ has a smallest element.*

Proof: Let $S \subseteq \mathbb{N}$ be non-empty. Because S is bounded below by zero, the infimum $x := \inf(S)$ exists (Proposition 1.1.16). Since this is the greatest lower bound, $x+1$ is **not** a lower bound on S. Hence, there exists $n \in S$ such that $x \leq n < x+1$. This implies that $x > n-1$. If n is **not** the least element, there exists $m \in S$ with $m < n$ which means $m \leq n-1 < x$, a contradiction to the fact that x is a lower bound on S. □

Theorem 1.2.5 (Density of $\mathbb{Q}$ in $\mathbb{R}$) *Between two distinct real numbers, there is a rational.*

Proof: Let $x, y \in \mathbb{R}$ such that $x < y$ without loss of generality. This implies $y - x > 0$, so Corollary 1.2.3 can be used to find $N \in \mathbb{N}$ with $0 < \frac{1}{N} < y - x$. We next define the subset

$$S := \left\{ n \in \mathbb{N} : \frac{n}{N} > x \right\} \subseteq \mathbb{N}.$$

This set is non-empty by Corollary 1.2.4. By the well-ordering of the naturals (Lemma 1.2.6), there exists a smallest element $M \in S$. By definition, this means $x < \frac{M}{N}$ and, since M is the smallest element, $M-1 \notin S$, that is $x \not< \frac{M-1}{N}$. This is equivalent to saying $\frac{M-1}{N} \leq x$. But now,

$$\frac{M}{N} = \frac{M-1}{N} + \frac{1}{N} < \frac{M-1}{N} + y - x \leq x + y - x = y.$$

Combining things together, we have $x < \frac{M}{N} < y$, and clearly $\frac{M}{N} \in \mathbb{Q}$ is a rational number. □

1.3 The Absolute Value

Definition Let $x \in \mathbb{R}$. The absolute value of x is the real number $|x| \in \mathbb{R}$ defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

Note: We can also define $|x| = \max\{x, -x\}$, from which we see $x \leq |x|$ and $-x \leq |x|$.

Theorem 1.3.1 *For all $x, y \in \mathbb{R}$, the absolute value has the following properties:*

(i) $|x| \geq 0$ *with equality if and only if* $x = 0$. (**Non-Negativity**)
(ii) $|-x| = |x|$. (**Evenness**)
(iii) $|xy| = |x||y|$. (**Multiplicativity**)
(iv) $|x+y| \leq |x| + |y|$. (**Triangle Inequality**)
(v) $|x-y| \geq ||x| - |y||$. (**Reverse Triangle Inequality**)

Proof: (i) This is immediate from the definition of the absolute value.

(ii) This is also immediate from the definition of the absolute value.

(iii) Multiplicativity can be checked by considering the four cases of possible signs for x and y: $x > 0$ and $y > 0$; $x > 0$ and $y < 0$; $x < 0$ and $y > 0$; $x < 0$ and $y < 0$. Of course, if any of x and y are zero, the result is trivial.

(iv) We know from the above note that $x \leq |x|$ and $y \leq |y|$; adding these inequalities tells us that $x + y \leq |x| + |y|$. Similarly, $-x \leq |x|$ and $-y \leq |y|$; adding these inequalities gives us $-(x + y) \leq |x| + |y|$. Combining these statements is precisely that $|x + y| \leq |x| + |y|$.

(v) This follows from the usual Triangle Inequality in the following context:

$$|y| = |x + (y - x)| \leq |x| + |y - x| \qquad \text{and} \qquad |x| = |(x - y) + y| \leq |x - y| + |y|.$$

The first of the above inequalities implies that $|y| - |x| \leq |y - x|$, and the second implies $|x| - |y| \leq |x - y|$. But the right-hand sides of each of these inequalities are equal by part (ii). Since the left-hand sides are opposite signs of each other, we can also apply part (ii) to conclude that

$$||x| - |y|| \leq |x - y|. \qquad \square$$

Note: We can replace y with $-y$ to get an equivalent inequality to the Triangle Inequality:

$$|x - y| \leq |x| + |y|.$$

This implies the following inequality, for all $x, y, z \in \mathbb{R}$:

$$|x - y| \leq |x - z| + |y - z|.$$

2 Sequences and Convergence

2.1 Sequences of Real Numbers

> **Definition** A sequence of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$ where we denote the output a_n (instead of the usual notation $a(n)$ for functions). A term in the sequence is denoted by a_n, whereas the whole sequence is denoted by $(a_n)_{n\in\mathbb{N}}$, or just (a_n) for short.

In practice, we think of a real sequence as an infinite ordered list $(a_n) = (a_1, a_2, a_3, ...)$ of numbers.

2.2 Convergence of Sequences

> **Definition 2.2.3** A real sequence (a_n) converges to a real number $L \in \mathbb{R}$ if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. In this case, we call L the limit of (a_n) and we write either $a_n \to L$ or $\lim_{n\to\infty} a_n = L$. Here, we call (a_n) convergent.

Remark Let's take a breather; Definition 2.2.3 is the first rigorous definition of a limit we have encountered, and later definitions will have a similar flavour. Therefore, it is important you have an idea of what this definition says. We will try to stimulate this intuition now.

Given a sequence (a_n), we can show that it 'approaches' the number L as $n \in \mathbb{Z}^+$ gets large by showing that for **any** positive number ($\varepsilon > 0$), there exists a point in the sequence a_N (there exists $N \in \mathbb{Z}^+$) for which it and every subsequent term in the sequence (for all $n \geq N$) lies within distance that positive number of the number L ($|a_n - L| < \varepsilon$). Because this needs to work for **any** ε, the idea is that the distance can be as large or as small as you like and we should still be able to find $N \in \mathbb{Z}^+$ to make this work. Geometrically, if we plot n against a_n, every point for $n \geq N$ will live inside a rectangle with width 2ε centred on the line $a_n = L$; see Figure 1 below.

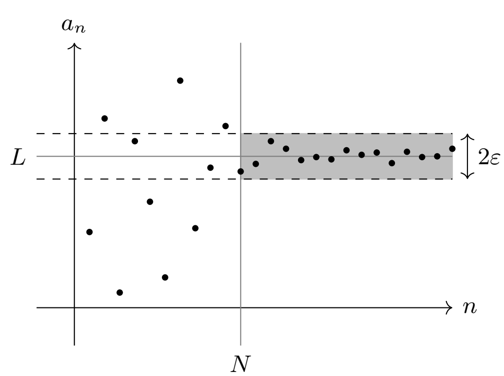

Figure 1: The geometric interpretation of the convergence of a real sequence (a_n).

> **Note:** Thus, (a_n) converges if and only if there are a **finitely-many** $a_n \notin (L - \varepsilon, L + \varepsilon)$.

Theorem 2.2.6 (Uniqueness of Limits) *The limit of a convergent sequence is unique.*

Proof: Let (a_n) be convergent and suppose that $a_n \to L$ and $a_n \to K$. We must prove $L = K$. Indeed, suppose $\varepsilon > 0$ is given. By Definition 2.2.3 applied to each of these convergences in turn, there exist $N_1, N_2 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|a_n - L| < \frac{\varepsilon}{2}$ and, for all $n \geq N_2$, $|a_n - K| < \frac{\varepsilon}{2}$. Let's now pick $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, we see that

$$\begin{aligned}
|L - K| &= |L - a_n + a_n - K| \\
&\leq |L - a_n| + |a_n - K|, && \text{by the Triangle Inequality,} \\
&= |a_n - L| + |a_n - K|, && \text{by properties of the absolute value,} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, && \text{by the inequalities above,} \\
&= \varepsilon.
\end{aligned}$$

This shows that the 'distance' between the real numbers L and K is less than the positive number ε, but this works for **any** ε, so we must have $|L - K| = 0$. In other words, $L = K$ as required. □

Definition 2.2.8 A sequence (a_n) is constant if there exists $c \in \mathbb{R}$ with $a_n = c$ for all n.

Proposition 2.2.9 *A constant sequence $a_n \equiv c \to c$.*

Proof: Let $\varepsilon > 0$ be given and (a_n) be a constant sequence whereby $a_n = c$ for all n. Then, for all n (in particular, $n \geq N$ for **any** choice of $N \in \mathbb{N}$), we see that $|a_n - c| = |c - c| = 0 < \varepsilon$. □

Proposition 2.2.10 *A sequence $a_n \to L$ if and only if the sequence $a_n - L \to 0$.*

Proof: This is basically a tautology, but nevertheless we argue in full. Let $\varepsilon > 0$ be given. Then,

$$\begin{aligned}
& a_n \to L \\
\Leftrightarrow \quad & \text{there exists } N \in \mathbb{N} \text{ such that, for all } n \geq N, \text{ we have } |a_n - L| < \varepsilon \\
\Leftrightarrow \quad & \text{there exists } N \in \mathbb{N} \text{ such that, for all } n \geq N, \text{ we have } |(a_n - L) - 0| < \varepsilon \\
\Leftrightarrow \quad & a_n - L \to 0.
\end{aligned}$$

□

Proposition 2.2.11 (Dominating Sequences) *Let (a_n) and (b_n) be sequences with $b_n \to 0$ and, for some $N \in \mathbb{N}$, $|a_n| \leq |b_n|$ for every $n \geq N$. Then, $a_n \to 0$.*

Proof: By definition, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|b_n| < \varepsilon$ for all $n \geq K$. But then, we know that $|a_n| \leq |b_n| < \varepsilon$ for all $n \geq \max\{N, K\}$, which is precisely to say $a_n \to 0$. □

Theorem 2.2.12 (Squeeze Theorem) *Let $(a_n), (b_n), (c_n)$ be sequences where $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $a_n \to L$ and $c_n \to L$, then $b_n \to L$.*

Proof: By assumption, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|a_n - L| < \varepsilon$ and $|c_n - L| < \varepsilon$ (we can use the same N; if not, we can choose N to be the larger of the two naturals that ensures each of these inequalities holds). By the properties of the absolute value, this means in particular that $L - \varepsilon < a_n$ and $c_n < L + \varepsilon$. Consequently, for all $n \geq N$, we have

$$L - \varepsilon < a_n \leq b_n \quad \text{and} \quad b_n \leq c_n < L + \varepsilon \qquad \Rightarrow \qquad L - \varepsilon < b_n < L + \varepsilon.$$

This is equivalent to $-\varepsilon < b_n - L < \varepsilon$, that is $|b_n - L| < \varepsilon$. Consequently, $b_n \to L$ as required. □

2.3 Boundedness and Monotonicity

Definition 2.3.1 Let (a_n) be a sequence.

(i) It is bounded above if there exists $M \in \mathbb{R}$ where $a_n \leq M$ for all $n \in \mathbb{N}$. In this case, we call the number M an upper bound for the sequence.

(ii) It is bounded below if there exists $K \in \mathbb{R}$ where $a_n \geq K$ for all $n \in \mathbb{N}$. In this case, we call the number K a lower bound for the sequence.

(iii) It is bounded if it is both bounded above and below.

Note: Boundedness of sequences can be rephrased in terms of boundedness of sets (from Definition 1.1.7) by instead considering the subset $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$.

Proposition 2.3.2 *A sequence (a_n) is bounded if and only if there exists $M \in \mathbb{R}$ such that*

$$|a_n| \leq M \text{ for all } n.$$

Proof: If (a_n) is bounded, it means it is bounded below (by K, say) and above (by L, say), that is $K \leq a_n \leq L$ for all n. But it is clear that $|a_n| \leq \max\{|K|, |L|\} =: M$. Conversely, if $|a_n| \leq M$, this is equivalent to $-M \leq a_n \leq M$. Hence, $-M$ is a lower bound and M is an upper bound. □

Proposition 2.3.3 *If (a_n) is bounded and $b_n \to 0$, then the sequence $c_n := a_n b_n \to 0$.*

Proof: By Proposition 2.3.2, there exists $M \in \mathbb{N}$ such that $|a_n| \leq M$ for all n. Because $b_n \to 0$, for any $\varepsilon > 0$ we can find some $N \in \mathbb{N}$ such that, for all $n \geq N$, $|b_n| < \frac{\varepsilon}{M}$. But for all $n \geq N$,

$$|c_n| = |a_n b_n| = |a_n||b_n| \leq M|b_n| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

□

Theorem 2.3.4 *Any convergent sequence is bounded.*

Proof: Suppose $a_n \to L$. Then, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $|a_n - L| < 1$ (remember this works for **all** $\varepsilon > 0$, so it works for $\varepsilon = 1$ in particular). By the Triangle Inequality, this means $|a_n| \leq |L| + 1$ for all $n \geq N$. In other words, $\{|a_n| : n \geq N\}$ is bounded (by $|L| + 1$), but we aren't done here. It might be that an earlier term of the sequence is larger than this number; we consider $\{|a_n| : n < N\}$ and therefore define $K \coloneqq \max(\{|a_n| : n < N\} \cup \{|L| + 1\})$. From this, it is now clear that the sequence satisfies $|a_n| \leq K$ for all n. □

Lemma 2.3.5 *If $a_n \to L$ with $a_n \neq 0$ for all $n \in \mathbb{N}$ and $L \neq 0$, then $(\frac{1}{a_n})$ is bounded.*

Proof: The structure of the argument here is similar to that of Theorem 2.3.4, namely we show $\{|\frac{1}{a_n}| : n \geq N\}$ is bounded, and then take the maximum of this and the previous terms. Since $a_n \to L$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|a_n - L| < \frac{|L|}{2}$ (choosing $\varepsilon = \frac{|L|}{2}$ here). So,

$$-\frac{|L|}{2} < a_n < L + \frac{|L|}{2} \qquad \Leftrightarrow \qquad L - \frac{|L|}{2} < a_n < L + \frac{|L|}{2}.$$

We want $\left|\frac{1}{a_n}\right| \leq K$ for some $K \in \mathbb{R}$. Using the above alongside the Triangle Inelasticity, we get

$$|L| = |a_n - L + a_n| \leq |a_n - L| + |a_n| < \frac{|L|}{2} + |a_n| \qquad \Rightarrow \qquad \frac{|L|}{2} < |a_n| \quad \Leftrightarrow \quad \left|\frac{1}{a_n}\right| < \frac{2}{|L|}.$$

We therefore choose $K \coloneqq \max(\{|\frac{1}{a_n}| : n < N\} \cup \{\frac{2}{|L|}\})$, from which we see $\left|\frac{1}{a_n}\right| \leq K$ for all n. □

Theorem 2.2.7 (Algebra of Limits) *Let $a_n \to A$ and $b_n \to B$. Then, the following are true:*

(i) $a_n + b_n \to A + B$.

(ii) $ca_n \to cA$ *for all* $c \in \mathbb{R}$.

(iii) $a_n b_n \to AB$.

(iv) $a_n / b_n \to A/B$ **if** $b_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $B \neq 0$.

Proof: (i) For each $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|a_n - A| < \frac{\varepsilon}{2}$ and, for all $n \geq N_2$, $|b_n - B| < \frac{\varepsilon}{2}$. Setting $N = \max\{N_1, N_2\}$, we see that for all $n \geq N$,

$$\begin{aligned} |a_n + b_n - (A + B)| &= |a_n - A + b_n - B| \\ &\leq |a_n - A| + |b_n - B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

(ii) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ where, for all $n \geq N$, $|a_n - A| < \frac{\varepsilon}{|c|}$. For all $n \geq N$ then,

$$|ca_n - aA| = |c(a_n - A)| = |c||a_n - a| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

(iii) Consider the sequence

$$a_nb_n - AB = A(b_n - B) + b_n(a_n - A).$$

Because $a_n \to A$ and $b_n \to B$, Proposition 2.2.10 implies that $a_n - A \to 0$ and $b_n - B \to 0$. Now, the sequence $(A(b_n - B))$ is a constant times a convergent sequence, so part (ii) of the Algebra of Limits tells us that $A(b_n - B) \to A \cdot 0 = 0$. Next, the sequence $(b_n(a_n - A))$ is the product of two convergent sequences, one of which converges to zero. In particular, it is the product of a bounded sequence (by Theorem 2.3.4) and sequence that converges to zero, so Proposition 2.3.3 implies that $b_n(a_n - A) \to 0$. Hence, part (i) of the Algebra of Limits implies $a_nb_n - AB \to 0$, which is equivalent to $a_nb_n \to AB$.

(iv) Consider the sequence

$$\frac{1}{b_n} - \frac{1}{B} = \frac{b - b_n}{Bb_n} = -\frac{1}{B}(\frac{1}{b_n}(b_n - B)).$$

Because $b_n \to B$, Proposition 2.2.10 implies $b_n - B \to 0$, and we know that $(\frac{1}{b_n})$ is bounded by Lemma 2.3.5. Hence, it follows that $\frac{1}{b_n} - \frac{1}{B} \to -\frac{1}{B} \cdot 0 = 0$, which is equivalent to $\frac{1}{b_n} \to \frac{1}{B}$. □

Note: One can give direct ε-N proofs for parts (iii) and (iv) of the Algebra of Limits.

ε-N Proofs of the Algebra of Limits: (iii) Since (b_n) is convergent, it is bounded by Theorem 2.3.4; there exists $M \in \mathbb{R}$ such that $|b_n| \leq M$ by Proposition 2.3.2. Let $\varepsilon > 0$ be given and define

$$\delta := \frac{\varepsilon}{M + |A|} > 0.$$

There exist $N_1, N_2 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|a_n - A| < \delta$ and, for all $n \geq N_2$, $|b_n - B| < \delta$. Consequently, for all $n \geq N$ where we have chosen $N = \max\{N_1, N_2\}$, it follows that

$$\begin{aligned} |a_nb_n - AB| &= |a_nb_n - Ab_n + Ab_n - AB| \\ &= |(a_n - A)b_n - A(b_n - B)| \\ &\leq |a_n - A||b_n| + |A||b_n - B| \\ &< \delta K + |A|\delta \\ &= \varepsilon. \end{aligned}$$

(iv) It suffices to prove $\frac{1}{b_n} \to \frac{1}{B}$; the result follows from part (ii). Let $\varepsilon > 0$ be given and define

$$\delta := \varepsilon\frac{|B|^2}{2} > 0$$

We know $\frac{|B|}{2} > 0$ as $B \neq 0$. Now, there exist $N_1, N_2 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|b_n - B| < \frac{|B|}{2}$ and, for all $n \geq N_2$, $|b_n - B| < \delta$. Again, setting $N = \max\{N_1, N_2\}$, we see that for all $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|b_n - B|}{|b_n||B|} < \frac{\delta}{|B||B|/2} = \varepsilon.$$ □

Method – Proofs without the Algebra of Limits: Suppose we are asked to prove that a product of two sequences converges **without** using the Algebra of Limits. Then, we can just 'copy' the proof of Theorem 2.2.7 with our sequences/limits substituted into it.

Definition 2.3.6 Let (a_n) be a sequence.
(i) It is (monotonically) increasing if $a_n \leq a_{n+1}$ for all n.
(ii) It is (monotonically) decreasing if $a_n \geq a_{n+1}$ for all n.
(iii) It is strictly (monotonically) increasing if $a_n < a_{n+1}$ for all n.
(iv) It is strictly (monotonically) decreasing if $a_n > a_{n+1}$ for all n.
(v) It is monotonic of it is either increasing or decreasing.

Theorem 2.3.7 (Monotone Convergence Theorem) *Bounded monotone sequences converge:*
(i) *Increasing sequences (a_n) that are bounded above converge to $\sup\{a_n : n \in \mathbb{N}\}$.*
(ii) *Decreasing sequences (a_n) that are bounded below converge to $\inf\{a_n : n \in \mathbb{N}\}$.*

Proof: (i) By the Axiom of Completeness, we know that $L := \sup\{a_n : n \in \mathbb{N}\}$ exists. Let $\varepsilon > 0$ be given. Because L is the *least* upper bound, $L - \varepsilon < L$ is not an upper bound; there is at least one term in the sequence larger than it, i.e. there exists $N \in \mathbb{N}$ such that $a_N > L - \varepsilon$. Because (a_n) is increasing, we know that $a_n \geq a_N$ for all $n \geq N$; this means that $a_n > L - \varepsilon$ for all $n \geq N$. Combining this with the fact that $a_n \leq L$ because L is an upper bound on the sequence, it follows that $L - \varepsilon < a_n < L + \varepsilon$ for all $n \geq N$, which is precisely to say $|a_n - L| < \varepsilon$.

(ii) If (a_n) is decreasing, then $b_n := -a_n$ is increasing, so we can simply apply part (i) to (b_n). □

Definition 2.3.8 Let $S \subseteq \mathbb{R}$. We say a sequence (a_n) is contained in S if $a_n \in S$ for all n.

Note: If $a_n \to L$ and (a_n) is contained in S, it is **not** true *in general* that the limit $L \in S$.

Theorem 2.3.9 *Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. Then, there exists a sequence (a_n) in S such that $a_n \to \sup(S)$.*

Proof: Because $L := \sup(S)$ is the *least* upper bound on S, the number $L - \frac{1}{n}$ is **not** an upper bound on S. This implies there exists an element $a_n \in S$ also living in the interval $(L - \frac{1}{n}, L]$. Doing this for all $n \in \mathbb{N}$, we obtain a sequence (a_n). Notice that $L - \frac{1}{n} < a_n \leq L$ for all n, so the Squeeze Theorem implies $a_n \to L$. □

Corollary 2.3.10 *Let $S \subseteq \mathbb{R}$ be non-empty and bounded below. Then, there exists a sequence (a_n) in S such that $a_n \to \inf(S)$.*

Sketch of Proof: Define $b_n := -a_n$ and apply Theorem 2.3.9 to (b_n). □

Definition 2.3.11 We call $S \subseteq \mathbb{R}$ closed if the limit of a convergent sequence in S lies in S.

Lemma 2.3.13 (Stability of Closed Inequalities under Limits) *Let $a_n \to A$ and $b_n \to B$ be sequences such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then, their limits satisfy $A \leq B$.*

Proof: Assume to the contrary that $A > B$, and let $\varepsilon = \frac{A-B}{2} > 0$. By the convergence of the sequences (a_n) and (b_n), there exist $N_1, N_2 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|a_n - A| < \varepsilon$ and, for all $n \geq N_2$, $|b_n - B| < \varepsilon$. In particular, this means that for all $n \geq \max\{N_1, N_2\}$, we have both

$$A - \varepsilon < a_n \qquad \text{and} \qquad b_n < B + \varepsilon.$$

Using the expression of ε that we selected at the beginning, one can see that

$$a_n > A - \varepsilon = A - \frac{A-B}{2} = \frac{A+B}{2} = B + \frac{A-B}{2} = B + \varepsilon > b_n,$$

contradicting the fact that $a_n \leq b_n$ for all n. □

Note: If we replace the closed inequality $\leq$ with a strict inequality $<$ in Lemma 2.3.13, the result would **not** be true. As an easy example, notice $-\frac{1}{n} < \frac{1}{n}$ but their limits $0 \not< 0$.

Proposition 2.3.12 *Closed intervals $[a, b]$ are closed in the sense of* Definition 2.3.11.

Proof: Let (c_n) be a convergent sequence contained in $[a, b]$. We must show that its limit L, let's call it, also lies in $[a, b]$. Well, the fact the sequence is contained in this interval means $a \leq c_n \leq b$ for all n. Thus, applying Lemma 2.3.13 tells us that $a \leq L \leq b$, that is $L \in [a, b]$. □

3 Subsequences

3.1 Definition and Convergence Properties

Definition 3.1.1 A sequence (b_k) is called a subsequence of (a_n) if there exists a strictly increasing sequence of positive integers (n_k) such that $b_k = a_{n_k}$ for all $k \in \mathbb{N}$.

Note: In other words, the terms in (b_k) must occur in (a_n) in the same order. An alternate take is this: we can obtain (b_k) from (a_n) by deleting possibly infinitely-many terms.

Theorem 3.1.3 *If $a_n \to L$ and (b_k) is a subsequence of (a_n), then $b_k \to L$.*

Proof: Let $\varepsilon > 0$ be given. Since $a_n \to L$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. By the definition of a subsequence, we know that $b_k = a_{n_k}$, where (n_k) is a strictly increasing sequence of positive integers. Note that $n_1 \geq 1$ and, if $n_k \geq k$, then $n_{k+1} \geq n_k + 1 \geq k + 1$. By induction, we conclude that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore, for all $k \geq N$, we have $n_k \geq n_N \geq N$ which means $|b_k - L| = |a_{n_k} - L| < \varepsilon$. □

Definition Let (a_n) be a sequence. We call a term a_m dominant if every subsequent term is **not** larger than it, that is to say $a_n \leq a_m$ for all $n > m$.

Lemma 3.1.5 *Every sequence has a monotonic subsequence.*

Proof: Let (a_n) be a sequence and D be the set of dominant terms.

(i) If D is infinite, the subsequence of dominant terms is decreasing, by definition of dominant; we have found a monotonic(ally decreasing) subsequence.

(ii) If D is finite (or empty), there exists a term a_m beyond which there are **no** dominant terms. Let $n_1 = m + 1$; since a_{n_1} is **not** dominant, there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$, but since a_{n_2} is **not** dominant, there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$, and so forth. This implies (a_{n_k}) is increasing; we have found a monotonic(ally increasing) subsequence. □

Theorem 3.1.6 (Bolzano-Weierstrass) *Any bounded sequence has a convergent subsequence.*

Proof: Let (a_n) be bounded, meaning $|a_n| \leq M$ for some $M \in \mathbb{N}$. Then, there exists a monotonic subsequence (a_{n_k}) by Lemma 3.1.5. It follows that (a_{n_k}) is also bounded by the same upper bound, that is $|a_{n_k}| \leq M$. Hence, (a_{n_k}) converges by the Monotone Convergence Theorem. □

3.2 The Cauchy Property

Definition 3.2.1 A sequence (a_n) is Cauchy (or has the Cauchy property) if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n, m \geq N$, we have $|a_n - a_m| < \varepsilon$.

Remark The definition of Cauchy is very similar to that of convergent, with a key difference; no mention of a real number L. Instead, we look at the difference between two terms a_n and a_m. In words, where convergence is about having all terms after a certain point being within distance ε of the limit L, the Cauchy property is about having all terms after a certain point being within distance ε of **each other**.

Proposition 3.2.2 *If (a_n) is convergent, then it is Cauchy.*

Proof: Suppose $a_n \to L$ and let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $|a_n - L| < \frac{\varepsilon}{2}$. But then, for all $n, m \geq N$, we have

$$\begin{aligned} |a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |a_m - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

Lemma 3.2.3 *If (a_n) is Cauchy, then it is bounded.*

Proof: The proof is similar to that of Theorem 2.3.4. By assumption, there exists $N \in \mathbb{N}$ such that, for all $n, m \geq N$, $|a_n - a_m| < 1$. By the Triangle Inequality, this means that $|a_n| \leq |a_N| + 1$ (since the first integer m which is at least N is N itself). We now only need to consider the maximum of the terms $|a_n|$ for $n < N$. Well, if we define $M := \max(\{|a_n| : n < N\} \cup \{|a_N| + 1\})$, we immediately see that $|a_n| \leq M$ for **all** n, which is precisely that (a_n) is bounded. □

Lemma 3.2.4 *If (a_n) is Cauchy and it has a subsequence $a_{n_k} \to L$, then $a_n \to L$.*

Proof: Let $\varepsilon > 0$ be given and consider both the convergence and the Cauchy property: there exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|a_{n_k} - L| < \frac{\varepsilon}{2}$ and there exists $N_2 \in \mathbb{N}$ such that, for all $n, m \geq N_2$, $|a_n - a_m| < \frac{\varepsilon}{2}$. By fixing $N = \max\{N_1, N_2\}$ and considering $n \geq N$, we have

$$\begin{aligned} |a_n - L| &= \left|a_n - a_{n_{N+1}} + a_{n_{N+1}} - L\right| \\ &\leq \left|a_n - a_{n_{N+1}}\right| + \left|a_{n_{N+1}} - L\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

Theorem 3.2.5 *A sequence (a_n) converges if and only if it is Cauchy.*

Proof: The forward direction is just Proposition 3.2.2. Conversely, if (a_n) is Cauchy, then it is bounded (Lemma 3.2.3). Hence, it has a convergent subsequence (Bolzano-Weierstrass Theorem), which means (a_n) itself converges (Lemma 3.2.4). □

4 Series

4.1 Definition and Convergence

Definition 4.1.1 A (real) series is a sequence of real numbers (s_k) with terms defined by

$$s_k = \sum_{n=1}^{k} a_n,$$

where a_n is the n^{th} summand and s_k is the k^{th} partial sum. The series is denoted $\sum_{n=1}^{\infty} a_n$.

Note: $\sum_{n=1}^{\infty} a_n$ is convergent if the sequence of partial sums (s_k) converges in the usual sense.

Proposition 4.1.2 (Divergence Test) *If $a_n \not\to 0$, then the series $\sum_{n=1}^{\infty} a_n$ does **not** converge.*

Proof: Suppose that $a_n \not\to 0$, meaning that there exists $\varepsilon > 0$ such that, for all $N \in \mathbb{N}$, there exists $n \geq N$ where $|a_n - 0| = |a_n| \geq \varepsilon$ (this is the negation of Definition 2.2.3). For any $N \in \mathbb{N}$ then, we can apply this assumption also to $N+1$, i.e. there exists $n+1 \geq N+1$ such that $|a_{n+1}| \geq \varepsilon$. Because we can write $s_{n+1} = s_n + a_{n+1}$, it follows that for all $n, n+1 \geq N$,

$$|s_{n+1} - s_n| = |a_{n+1}| \geq \varepsilon.$$

In other words, (s_k) is **not** Cauchy, and thus **not** convergent by Theorem 3.2.5. □

Lemma (Harmonic Series) *The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Proof: Per Theorem 3.1.3, it suffices to show the sequence of partial sums (s_k) has an unbounded (and thus divergent) subsequence. To that end, consider the subsequence (s_{2^p}) defined as follows:

$$\begin{aligned}
s_{2^p} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^{p-1}+1} + \cdots + \frac{1}{2^p} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{p-1}+1} + \cdots + \frac{1}{2^p}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^p} + \cdots + \frac{1}{2^p}\right) \\
&= 1 + \frac{p}{2}.
\end{aligned}$$

This is clearly unbounded, so the subsequence (s_{2^p}) diverges. Consequently, (s_k) diverges. □

Lemma (Geometric Series) *For $r \in \mathbb{R}$ with $|r| < 1$, the series $\sum\limits_{n=1}^{\infty} r^n$ converges to $\frac{1}{1-r}$.*

Proof: Consider the sequence of partial sums, in particular the k^{th} partial sum which is

$$\begin{aligned} && s_k &= 1 + r + r^2 + \cdots + r^k \\ \Rightarrow && rs_k &= r + r^2 + r^3 + \cdots + r^{k+1} \\ \Rightarrow && (1-r)s_k &= 1 - r^{k+1} \\ \Rightarrow && s_k &= \frac{1 - r^{k+1}}{1-r}. \end{aligned}$$

Applying the Algebra of Limits, noting that $r^{k+1} \to 0$, we conclude $s_k \to \frac{1}{1-r}$ as required. □

Remark It is usually not possible to derive such a nice formula for s_k as was done for the geometric series proof. In fact, it would be better to develop some tests based on the sequence (a_n) of terms in the series rather than the sequence (s_k) of partial sums; this would (will) make life a bit easier.

4.2 Convergence Tests for Series

Note: If **every** summand $a_n \geq 0$, then (s_k) is increasing as $s_{k+1} = s_k + a_{k+1} \geq s_k$. The Monotone Convergence Theorem implies the series converges if and only if (s_k) is bounded.

Proposition 4.2.1 (Algebra of Series) *Let $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converge. Then, we have these:*

(i) $\sum\limits_{n=1}^{\infty} (a_n + b_n) = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$.

(ii) $\sum\limits_{n=1}^{\infty} (ca_n) = c \sum\limits_{n=1}^{\infty} a_n$ *for all* $c \in \mathbb{R}$.

Sketch of Proof: This is an easy consequence of the Algebra of Limits for sequences. □

Lemma 4.2.2 *The series $\sum\limits_{n=1}^{\infty}$ converges if and only if $\sum\limits_{n=N}^{\infty} a_n$ converges for any $N \in \mathbb{N}$.*

Proof: Let (s_k) and (t_k) be the sequences of partial sums of $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=N}^{\infty} a_n$, respectively. So,

$$t_k = s_{k+(N-1)} - \sum_{n=1}^{N-1} a_n.$$

As (s_k) converges if and only if $(s_{k+(N-1)})$ converges, and the negative right-hand term is constant (which converges by Proposition 2.2.9), then (s_k) converges if and only if (t_k) converges. □

Theorem 4.2.3 (Comparison Test) *Let $a_n \geq 0$ and $b_n > 0$ for all $n \in \mathbb{N}$.*

(i) *If $(\frac{a_n}{b_n})$ is bounded above and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

(ii) *If $(\frac{b_n}{a_n})$ is bounded above and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Note: A less general (but perhaps more clear) statement of the Comparison Test is this: if we have $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} b_n$ converges implies that $\sum_{n=1}^{\infty} a_n$ converges.

Proof: (i) Let (s_k) and (t_k) be the sequences of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. By assumption,

$$0 \leq \frac{a_n}{b_n} \leq M \qquad \Leftrightarrow \qquad 0 \leq a_n \leq M b_n$$

for some $M \in \mathbb{R}$. Consequently, $s_k \leq M t_k$ for all k. Since we assume that (t_k) converges, it is bounded above by $K \in \mathbb{R}$, say. Therefore, $s_n \leq MK$. But now, (s_k) is monotonically increasing and bounded above, so converges by the Monotone Convergence Theorem.

(ii) This is merely the contrapositive of part (i). □

Method – Using the Comparison Test: Suppose we have a series and want to determine whether or not it converges. We ask ourselves "*what familiar series does this appear like which we* **do** *know converges or diverges?*", and this is precisely what we compare with.

Lemma *The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges .*

Sketch of Proof: Use the Comparison Test, comparing with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. □

Proposition 4.2.5 *The series $\sum_{n=1}^{\infty} \frac{1}{n^\ell}$ converges for all $\ell \in \mathbb{N}$ with $\ell \geq 2$.*

Proof: Use the Comparison Test, comparing with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Indeed, let

$$a_n = \frac{1}{n^\ell} \geq 0 \qquad \text{and} \qquad b_n = \frac{1}{n^2} > 0.$$

Then, we have $\frac{a_n}{b_n} = \frac{1}{n^{\ell-2}} \leq 1$, that is $(\frac{a_n}{b_n})$ is bounded above. Hence, the series converges. □

4.3 Absolute Convergence

Definition The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges as usual.

Theorem 4.3.1 *If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof: Let (s_k) and (t_k) be the sequences of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$. We see that

$$|t_m - t_k| = \sum_{n=k+1}^{m} |a_n|.$$

Because we assume (t_k) converges, it is Cauchy by Theorem 3.2.5: for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m > k \geq N$, we have $|t_m - t_k| < \varepsilon$. Therefore, for all $m > k \geq N$, we can apply the Triangle Inequality to the absolute value of the sum and use the above to conclude

$$|s_m - s_k| = \left| \sum_{n=k+1}^{m} a_n \right| \leq \sum_{n=k+1}^{m} |a_n| < \varepsilon. \qquad \square$$

Note: The converse statement fails, i.e. convergence does **not** imply absolute convergence. Indeed, a series that converges but **not** absolutely is called a conditionally convergent series.

Theorem 4.3.2 (Ratio Test) *Let $a_n > 0$ for all $n \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \to L$.*

(i) *If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and thus converges.*

(ii) *If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof: (i) Assume that $L < 1$. By assumption, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\left| \frac{a_{n+1}}{a_n} \right| < L + \frac{1-L}{2} =: r,$$

picking $\varepsilon = \frac{1-L}{2}$ in the definition of convergence. This inequality is equivalent to $|a_{n+1}| < r|a_n|$. Using induction, it follows that $|a_{N+m}| < r^m |a_N|$ for every $m \in \mathbb{N}$. If we take $b_n = r^n |a_N|$, the corresponding series converges since it is a geometric series multiplied by a constant. Hence, the Comparison Test tells us that the series in question is (absolutely) convergent.

(i) Assume that $L > 1$. By assumption, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\left| \frac{a_{n+1}}{a_n} \right| > L - \frac{L-1}{2} =: q,$$

picking $\varepsilon = \frac{L-1}{2}$ in the definition of convergence. This inequality is equivalent to $|a_{n+1}| > q|a_n|$. Using induction, it follows that $|a_{N+m}| > q^m|a_N|$ for every $m \in \mathbb{N}$. But $|a_{N+m}| > |a_N| > 0$ for all $m \in \mathbb{N}$, so $a_n \not\to 0$. By the Divergence Test, the series in question does **not** converge. $\square$

Note: If $\frac{a_{n+1}}{a_n} \to 1$, then we cannot immediately tell whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

4.4 Alternating Series

Definition 4.4.1 A series $\sum_{n=1}^{\infty} a_n$ is alternating if $a_n \neq 0$ and $\frac{a_{n+1}}{a_n} < 0$ for all $n \in \mathbb{N}$.

Lemma 4.4.2 *A series $\sum_{n=1}^{\infty} a_n$ is alternating if and only if it has the form $\pm \sum_{n=1}^{\infty} (-1)^{n+1}|a_n|$.*

Sketch of Proof: Use induction in two cases, depending on the sign of the first summand. $\square$

Lemma *Let (a_n) be a sequence and $L \in \mathbb{R}$. If $a_{2k} \to L$ and $a_{2k+1} \to L$, then $a_n \to L$.*

Proof: Let $\varepsilon > 0$ be given and apply Definition 2.2.3 to each of the subsequences in question:

- $a_{2k} \to L$ means there exists $K_1 \in \mathbb{N}$ such that, for all $k \geq K_1$, $|a_{2k} - L| < \varepsilon$.
- $a_{2k+1} \to L$ means there exists $K_2 \in \mathbb{N}$ such that, for all $k \geq K_2$, $|a_{2k+1} - L| < \varepsilon$.

Then, for all $n \geq \max\{2K_1, 2K_2+1\}$, either n is even (for which $n \geq 2K_1$ and so $|a_n - L| < \varepsilon$) or n is odd (for which $n \geq 2K_2+1$ and so $|a_n - L| < \varepsilon$). Either way, we conclude that $a_n \to L$. $\square$

Theorem 4.4.3 (Alternating Series Test) *Let (a_n) be a decreasing* **positive** *sequence which converges to zero. Then, the alternating series $\pm \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge.*

Proof: It suffices to work only with the positive series, as the negative case will follow from the Algebra of Series. Let (s_k) be the sequence of partial sums and consider the subsequence (s_{2m}):

$$\begin{aligned} s_{2m} &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2m-1} - a_{2m} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}) \\ \Rightarrow \qquad s_{2m+2} - s_{2m} &= a_{2m+1} - a_{2m+2} \\ &\geq 0. \end{aligned}$$

This implies that the sequence (s_{2m}) is increasing. Furthermore, we see that

$$s_{2m} = a_1 - (a_2 + a_3) - \cdots - (a_{2m-2} + a_{2m-1}) - a_{2m}$$

$$< a_1.$$

This establishes (s_{2m}) is bounded above by a_1. By the Monotone Convergence Theorem, we know that $s_{2k} \to L$, for some $L \in \mathbb{R}$. Consider now the subsequence (s_{2m+1}). But notice that $s_{2m+1} = s_{2m} + a_{2m+1}$. Applying the Algebra of Limits yields $s_{2m+1} \to L + 0 = L$, since we assume that $a_n \to 0$. Consequently, the previous lemma implies $s_k \to L$ and we are done. □

Corollary (Alternating Harmonic Series) *The series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converges.*

Proof: This is an immediate consequence of the Alternating Series Test with $a_n = \frac{1}{n}$. □

Note: This series actually acts as a counterexample to the **converse** of Theorem 4.3.1.

Definition 4.4.6 A series is conditionally convergent if it converges but **not** absolutely.

4.5 The Riemann Rearrangement Theorem (Non-Examinable)

Definition 4.5.1 For a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, we call $\sum_{n=1}^{\infty} a_{\sigma(n)}$ a rearrangement of $\sum_{n=1}^{\infty} a_n$.

Remark In other words, we have the same underlying sequence (a_n) of summands but they now appear in a different order. The fact that σ is a bijection means that each index n is sent to a single distinct new index $\sigma(n)$. Of course, if $\sigma(n) = n$, then we will have a *trivial rearrangement* wherein nothing has actually changed.

Note: The sequence (u_k) of partial sums of $\sum_{n=1}^{\infty} a_{\sigma(n)}$ **isn't** generally a subsequence of (s_k).

Theorem 4.5.2 *If* $\sum_{n=1}^{\infty} a_n$ *converges absolutely, then any rearrangement* $\sum_{n=1}^{\infty} a_{\sigma(n)}$ *converges.*

Proof: We begin by introducing the following sequences of partial sums that we use throughout:

$$(s_k) \text{ is the sequence of partial sums of } \sum_{n=1}^{\infty} a_n,$$

$$(t_k) \text{ is the sequence of partial sums of } \sum_{n=1}^{\infty} |a_n|,$$

$$(u_k) \text{ is the sequence of partial sums of } \sum_{n=1}^{\infty} a_{\sigma(n)}.$$

Because we assume that the series converges *absolutely*, we know there exists $M \in \mathbb{R}$ such that $t_k \to M$. This means that, for each $\varepsilon > 0$, there exists $K_1 \in \mathbb{N}$ such that, for all $k \geq K_1$,

$$|t_k - M| < \frac{\varepsilon}{4} \qquad \Leftrightarrow \qquad \left|\sum_{n=1}^{k} |a_n| - \sum_{n=1}^{\infty} |a_n|\right| = \sum_{n=k+1}^{\infty} |a_n| < \frac{\varepsilon}{4}.$$

But absolute convergence implies convergence (Theorem 4.3.1), so there exists $L \in \mathbb{R}$ such that $s_k \to L$. This means that, for each $\varepsilon > 0$, there exists $K_2 \in \mathbb{N}$ such that, for all $k \geq K_2$,

$$|s_k - L| < \frac{\varepsilon}{2} \qquad \Leftrightarrow \qquad L - \frac{\varepsilon}{2} < s_k < L + \frac{\varepsilon}{2}.$$

We now assume $k \geq K := \max\{K_1, K_2\}$ is large enough such that the first K summands $a_1, ..., a_K$ appear in the first k summands $a_{\sigma(n)}$ of the rearrangement. Thus, the difference of partial sums

$$u_k - s_k = \sum_{n=1}^{k} a_{\sigma(n)} - \sum_{n=1}^{k} a_n$$

has $a_1, ..., a_K$ appearing in each of the sums on the right, so these terms cancel. Consequently,

$$\begin{aligned}
|u_k - s_k| &= \left|\sum_{n=1}^{k} a_{\sigma(n)} - \sum_{n=1}^{k} a_n\right| \\
&= \left|\sum_{n=1,\sigma(n)\geq K+1}^{k} a_{\sigma(n)} - \sum_{n=K+1}^{k} a_n\right| \\
&\leq \left|\sum_{n=1,\sigma(n)\geq K+1}^{k} a_{\sigma(n)}\right| + \left|\sum_{n=K+1}^{k} a_n\right|, && \text{by the Triangle Inequality,} \\
&\leq \sum_{n=1,\sigma(n)\geq K+1}^{k} |a_{\sigma(n)}| + \sum_{n=K+1}^{k} |a_n|, && \text{by the Triangle Inequality,} \\
&\leq \sum_{j=K+1}^{\infty} |a_j| + \sum_{n=K+1}^{\infty} |a_n|, && \text{after setting } j := \sigma(n), \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4}.
\end{aligned}$$

This is equivalent to $s_k - \frac{\varepsilon}{4} < u_k < s_k + \frac{\varepsilon}{4}$. Combined with an earlier inequality, we see that

$$L - \varepsilon < s_k - \frac{\varepsilon}{2} < u_k < s_k + \frac{\varepsilon}{2} < L + \varepsilon \qquad \Leftrightarrow \qquad |u_k - L| < \varepsilon.$$

□

> **Note:** The proof shows the rearrangement converges to the **same limit** as the series, i.e.
>
> $$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n.$$

Theorem 4.5.3 (Riemann Rearrangement Theorem) *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be conditionally convergent and* $L \in \mathbb{R}$. *Then, there exists a rearrangement of this series which converges to* L, *i.e.*

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = L.$$

Sketch of Proof: Consider the following subsequences of (a_n), namely (b_ℓ) which consists of the non-negative terms, and (c_ℓ) which consists of the negative terms. Note each of these truly is a subsequence, i.e. there are infinitely-many non-negative and negative terms. Well, for all $N \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Leftrightarrow \sum_{n=N}^{\infty} a_n \text{ converges.} \qquad (*)$$

- If we have finitely-many **non-negative** terms, then for large enough N, we see that

$$\sum_{n=N}^{\infty} a_n = -\sum_{n=N}^{\infty} |a_n|.$$

- If we have finitely-many **negative** terms, then for large enough N, we see that

$$\sum_{n=N}^{\infty} a_n = -\sum_{n=N}^{\infty} |a_n|.$$

Convergence of the left-hand sides follows from the above equivalence $(*)$. Hence, it follows that the right-hand sides also converge, but $(*)$ tells us also that this means the series converges *absolutely*, contradicting the *conditionally* convergent hypothesis. We now consider the series generated by each of these subsequences. Let's call the limit of the original series $C \in \mathbb{R}$, that is

$$\sum_{n=1}^{\infty} a_n = C.$$

- Since $\sum\limits_{n=1}^{\infty} |a_n|$ diverges, for any $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, **both**

$$\sum_{n=1}^{k} a_n > C - 1 \qquad \text{and} \qquad \sum_{n=1}^{k} |a_n| > 2M - C + 1.$$

  The sequence of partial sums of $\sum\limits_{\ell=1}^{\infty} b_\ell$ is unbounded above (and doesn't converge). Indeed,

$$\sum_{\ell=1}^{k} b_\ell = \sum_{n=1, b_\ell = a_n}^{k} a_n = \frac{1}{2}\left(\sum_{n=1}^{k} a_n + \sum_{n=1}^{k} |a_n|\right) > M.$$

- Since $\sum_{n=1}^{\infty} |a_n|$ diverges, for any $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, **both**

$$\sum_{n=1}^{k} a_n < C + 1 \qquad \text{and} \qquad \sum_{n=1}^{k} |a_n| > -2M + C + 1.$$

The sequence of partial sums of $\sum_{\ell=1}^{\infty} c_\ell$ is unbounded below (and doesn't converge). Indeed,

$$\sum_{\ell=1}^{k} c_\ell = \sum_{n=1, c_\ell = a_n}^{k} a_n = \frac{1}{2}\left(\sum_{n=1}^{k} a_n - \sum_{n=1}^{k} |a_n|\right) < M.$$

Per $(*)$, the so-called *tail* of each of the series defined by these subsequences converge, meaning

$$\sum_{\ell=N}^{\infty} b_\ell \text{ converges} \qquad \text{and} \qquad \sum_{\ell=N}^{\infty} c_\ell \text{ converges}$$

for any $N \in \mathbb{N}$. By the Divergence Test, it is necessary that $a_n \to 0$, so Theorem 3.1.3 now tells us each of $b_\ell \to 0$ and $c_\ell \to 0$. For any $L \in \mathbb{R}$, we define the rearrangement we're after as follows:

1. Take as many summands b_ℓ as we can such that $\sum_{\ell=1}^{N-1} b_\ell < L$ but then $\sum_{\ell=1}^{N} b_\ell > L$.
2. Follow this by adding $c_1, ..., c_M$ so $\sum_{\ell=1}^{N} b_\ell + \sum_{\ell=1}^{M-1} c_\ell > L$ but then $\sum_{\ell=1}^{N} b_\ell + \sum_{\ell=1}^{M} c_\ell < L$.
3. Repeat this process continuously.

The idea is this: add some of the positive (or zero) (b_ℓ)-terms until we exceed L, and then add to this some of the negative (c_ℓ)-terms until we fall short of L, and add some more (b_ℓ)-terms until we again overshoot L, and so forth. This process will never fail because the series corresponding to each of these subsequences are unbounded above/below. Convergence to L is intuitively clear, but let's make it rigorous: for each $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that, for all $\ell \geq K$, we have $0 \leq b_\ell < \varepsilon$ (since $b_\ell \to 0$) and $-\varepsilon < c_n < 0$ (since $c_\ell \to 0$). The corresponding rearrangement is therefore bounded between $L + \varepsilon$ and $L - \varepsilon$, as required. □

> **Note:** Perhaps more remarkably, there exists a rearrangement that is a **divergent** series.

Justification: Continuing with what we established in the proof of Theorem 4.5.3, we can find a divergent rearrangement by taking as many (b_ℓ)-terms as we like to ensure that the sum is greater than two, and then as many (c_ℓ)-terms to bring it less than one, and then some more (b_ℓ)-terms so that it exceeds three, followed by (c_ℓ)-terms to get under two, and so forth. Eventually, the sum will exceed $M + 2$ for any $M \in \mathbb{R}$, but $c_\ell > -1$ for all $\ell \geq K$ and thus the partial sums are unbounded below (each is greater than M for K^{th} one and beyond). □

5 Symbolic Logic

Reminder: A statement is any declaration which is unambiguously either true or false.

5.1 Symbolic Manipulation

Definition Let P and Q be statements. We can build from these the following statements:

- The negation $\neg P$, which means "not P".
- The conjunction $P \wedge Q$, which means "P and Q".
- The disjunction $P \vee Q$, which means "P or Q or both".

Note: The conjunction is true precisely when **both** the statements P and Q are true. The disjunction is true precisely when **at least one** of the statements P and Q are true.

Definition A truth table encodes the truth/falsity of a constructed statement in terms of the truth/falsity of its constituent pieces. We represent "false" by 0 and "true" by 1.

Method – Logically-Equivalent Statements: To show that two statements are logically equivalent (meaning one is true if and only if the other is true), it suffices to construct a truth table and exhibit that the columns representing each of the statements are identical. In this case, we denote logical equivalence by the symbol $\Leftrightarrow$.

Lemma *For any statement P, we have the logical equivalence $P \Leftrightarrow \neg(\neg P)$.*

Proof: We construct the truth table and compare the relevant columns:

P	$\neg P$	$\neg(\neg P)$
0	1	0
1	0	1

□

Theorem (de Morgan's Laws) *For statements P and Q, we have these logical equivalences:*
(i) $\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$.
(ii) $\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$.

Proof: (i) We construct a truth table once again and compare the relevant columns:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

(ii) We could construct a truth table, or we can use what we have already proved. Indeed,

$$\begin{aligned}
\neg(P \vee Q) &\Leftrightarrow \neg(\neg(\neg P) \vee \neg(\neg Q)), && \text{by the previous lemma,} \\
&\Leftrightarrow \neg\big(\neg\big((\neg P) \wedge (\neg Q)\big)\big), && \text{by the first de Morgan's Law,} \\
&\Leftrightarrow (\neg P) \wedge (\neg Q), && \text{by the previous lemma again.}
\end{aligned}$$

□

5.2 Implications

Definition Let P and Q be statements. The implication $P \Rightarrow Q$ means "if P, then Q".

Note: The standard notation for an implication is actually $P \to Q$ but this conflicts with our notation of convergence, so we henceforth stick with the notation $P \Rightarrow Q$.

The implication is delicate when it comes to truth. Namely, it only says "if P is true, then Q is true automatically". Therefore, P being false doesn't contradict this statement, in which case it's true by default. The only instance where the implication is false is if P is true but Q is **not**:

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Lemma *For statements P and Q, we have the logical equivalence $(P \Rightarrow Q) \Leftrightarrow (\neg P) \vee Q$.*

Sketch of Proof: Simply construct the truth table for $(\neg P) \vee Q$. □

Definition Let P and Q be statements. We can build from these the following statements:
- The converse of $P \Rightarrow Q$, which is $Q \Rightarrow P$.
- The contrapositive of $P \Rightarrow Q$, which is $\neg Q \Rightarrow \neg P$.

Lemma *For statements P and Q, we have the logical equivalence $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$.*

Sketch of Proof: Simply construct the truth table for $\neg Q \Rightarrow \neg P$. □

Note: The negation of an implication is $\neg(P \Rightarrow Q) \Leftrightarrow \neg((\neg P) \vee Q) \Leftrightarrow P \wedge (\neg Q)$ by using the lemma-before-last. Hence, the negation of an implication is **not** itself an implication.

5.3 Quantifiers

Definition Let A be a set and $P(x)$ be a collection of statements indexed by $x \in A$.

- A universal statement is $\forall x\, P(x)$, which means "$P(x)$ is true for **all** $x \in A$".
- An existential statement is $\exists x\, P(x)$, which means "$P(x)$ is true for **some** $x \in A$".

Note: The symbol $\forall$ is the universal quantifier, and the symbol $\exists$ is the existential quantifier.

Proposition *Let $P(x)$ be true for all $x \in A$. Then, we have the following:*

(i) $\neg(\forall x\, P(x)) \Leftrightarrow \exists x\, \neg P(x)$.

(ii) $\neg(\exists x\, P(x)) \Leftrightarrow \forall x\, \neg P(x)$.

Sketch of Proof: These can be justified by thinking about negation in words. For (i), the statement $\neg(\forall x\, P(x))$ means it is **not** the case that $P(x)$ is true for **all** $x \in A$. In particular, there exists an element of A making it not true. Similarly for (ii), the statement $\neg(\exists x\, P(x))$ means it is **not** the case that $P(x)$ is true for some $x \in A$. In other words, every element of A makes the statement false. □

Theorem *The statement "the sequence (a_n) does* **not** *converge to $L \in \mathbb{R}$" means that there exists $\varepsilon > 0$ such that, for all $N \in \mathbb{N}$, there exists $n \geq N$ with $|a_n - L| \geq \varepsilon$.*

Sketch of Proof: Write out the definition of $a_n \to L$ using quantifiers and negate it. □

Corollary *The statement "the sequence (a_n) does* **not** *converge" means for all $L \in \mathbb{R}$, there exists $\varepsilon > 0$ such that, for all $N \in \mathbb{N}$, there exists $n \geq N$ with $|a_n - L| \geq \varepsilon$.*

Proof: The fact (a_n) converges means there exists $L \in \mathbb{R}$ such that Definition 2.2.3 is satisfied. Negating and using the above theorem and previous proposition, we get the expected result. □

6 Functions of a Real Variable

6.1 Continuity

Definition 6.1.1 Let $D \subseteq \mathbb{R}$. Then, $f : D \to \mathbb{R}$ is continuous at $a \in D$ if, for **all** sequences (x_n) in D where $x_n \to a$, we have $f(x_n) \to f(a)$. If f is **not** continuous at $a \in D$, we say it is discontinuous at $a \in D$. We say f is continuous if it is continuous at **every** $a \in D$.

Lemma *The constant function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) \equiv c$ for a fixed $c \in \mathbb{R}$ is continuous.*

Proof: Let $a \in \mathbb{R}$ and $x_n \to a$. Then, $f(x_n) = c \to c = f(a)$. □

Lemma *The identity function $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x$ is continuous.*

Proof: Let $a \in \mathbb{R}$ and $x_n \to a$. Then, $g(x_n) = x_n \to a = g(a)$. □

Lemma *The reciprocal function $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ where $h(x) = \frac{1}{x}$ is continuous.*

Proof: Let $a \in \mathbb{R} \setminus \{0\}$ and $x_n \to a$. Then, $h(x_n) = \frac{1}{x_n} \to \frac{1}{a} = h(a)$ by the Algebra of Limits. □

Theorem 6.1.3 (Algebra of Continuous Functions) *Let $f, g : D \to \mathbb{R}$ be continuous at $a \in D$. Then, the following are true:*

(i) *$f + g$ is continuous at a.*
(ii) *cf is continuous at a for all $c \in \mathbb{R}$.*
(iii) *fg is continuous at a.*
(iv) *$1/f$ is continuous at a* **if** *$f(x) \neq 0$ for all $x \in D$.*

Proof: Let (x_n) be a sequence in D such that $x_n \to a$. By assumption, we know $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$. Consequently, the usual Algebra of Limits of sequences implies the result:

(i) $(f + g)(x_n) = f(x_n) + g(x_n) \to f(a) + g(a) = (f + g)(a)$.

(ii) $(cf)(x_n) = cf(x_n) \to cf(a) = (cf)(a)$.

(iii) $(fg)(x_n) = f(x_n)g(x_n) \to f(a)g(a) = (fg)(a)$.

(iv) $(1/f)(x_n) = 1/f(x_n) \to 1/f(a) = (1/f)(a)$. □

Reminder: A polynomial in x is a function of the form $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ where the coefficients $a_0, ..., a_n \in \mathbb{R}$. If $a_n \neq 0$, we say that the degree of p is n, denoted $\deg(p)$.

Corollary 6.1.4 *Every polynomial function $p : \mathbb{R} \to \mathbb{R}$ is continuous.*

Proof: We prove this by induction on the degree of the polynomial, namely n. Indeed, the base case is where $n = 0$, so the polynomial is constant and is therefore continuous. Next, assume that every polynomial of degree k is continuous and let p be a polynomial of degree $k + 1$. Hence, we can write $p(x) = xq(x) + a_0$, where q is a polynomial of degree k. By assumption, q is continuous; we know that the polynomial $g(x) = x$ is continuous from earlier. Hence, the Algebra Property of Continuous Functions implies that p is continuous. By induction, the result holds for all polynomials. □

Corollary 6.1.5 *Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial and define $D = \{x \in \mathbb{R} : p(x) \neq 0\}$. Then, the function $f : D \to \mathbb{R}$ given by $f(x) = \dfrac{1}{p(x)}$ is continuous.*

Proof: Apply Theorem 6.1.3(iv) in conjunction with Corollary 6.1.4. □

Theorem 6.1.6 *Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ with $f(D) \subseteq E$ and assume f is continuous at $a \in D$ and g is continuous at $f(a) \in E$. Then, their composition $g \circ f : D \to \mathbb{R}$ given by $(g \circ f)(x) = g(f(x))$ is continuous at a.*

Proof: Let (x_n) be a sequence in D such that $x_n \to a$. Since f is continuous at a, $f(x_n) \to f(a)$. But since g is continuous at $f(a)$, $(g \circ f)(x_n) = g(f(x_n)) \to g(f(a)) = (g \circ f)(a)$, as required. □

Corollary 6.1.7 *Let p and q be polynomials and define $D = \{x \in \mathbb{R} : q(x) \neq 0\}$. Then, the rational function $f : D \to \mathbb{R}$ given by $f(x) = \dfrac{p(x)}{q(x)}$ is continuous.*

Proof: Combine Theorem 6.1.3, Corollaries 6.1.4 and 6.1.5, and Theorem 6.1.6. □

Note: Where the coefficients live in $\mathbb{R}$, we denote the set of polynomials in one variable by $\mathbb{R}[x]$. Furthermore, we denote the set of rational functions in one variable by $\mathbb{R}(x)$.

6.2 Limits of Functions

Definition 6.2.3 Let $D \subseteq \mathbb{R}$. We say $a \in D$ is a cluster point (or a limit point) of D if there exists a sequence (x_n) in $D \setminus \{a\}$ such that $x_n \to a$.

In other words, a cluster point is an element of a subset that can be approximated by a sequence of other elements of that subset. More informally, we can "tend to" $a \in D$ without reaching a.

Definition 6.2.1 Let $f : D \to \mathbb{R}$ and $a \in D$ a cluster point. We say f has a limit $L \in \mathbb{R}$ at a if for any (x_n) in $D \setminus \{a\}$ with $x_n \to a$, we have $f(x_n) \to L$. This is denoted $\lim_{x \to a} f(x) = L$.

Note: The value $f(a)$, if it even exists, has no bearing on the limit L of f at a (in general).

That said, there is a special connection between continuous functions and functions whose limit at a point is equal to the *value* of the function $f(a)$ at said point. This is explored below.

Proposition 6.2.4 (Continuity via Limits) *Let $f : D \to \mathbb{R}$ and $a \in D$ be a cluster point. Then, f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.*

Proof: ($\Rightarrow$) If f is continuous at a, then for all (x_n) in D such that $x_n \to a$, we have $f(x_n) \to f(a)$ by Definition 6.1.1. In particular then, this is true for all (x_n) in $D \setminus \{a\} \subseteq D$, so Definition 6.2.1 is satisfied with $L = f(a)$, that is $\lim_{x \to a} f(x) = f(a)$.

($\Leftarrow$) Let $f(a)$ be the limit of f at a and suppose (x_n) is any sequence in D such that $x_n \to a$. We will define the collection of numbers (b_k) simply be removing the terms $x_{n_k} = a$. There are two cases to consider:

- If (b_k) is finite, it is not itself a sequence, but $x_n = a$ for all $n \geq \max\{n_k : x_{n_k} \neq a\}$ and thus $f(x_n) \to f(a)$.
- If (b_k) is infinite, then it is a subsequence of (x_n) and it lives exclusively in $D \setminus \{a\}$. By Theorem 3.1.3, it follows that $b_k \to a$. Therefore, $f(b_k) \to f(a)$ by assumption; for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $k \geq N$, we have $|f(b_k) - f(a)| < \varepsilon$. If we now let $n \geq n_N$, either $x_n = b_k$ or $x_n = a$. Hence, $|f(x_n) - f(a)| < \varepsilon$, that is $f(x_n) \to f(a)$. $\square$

Proposition 6.2.6 *Let $f : D \to \mathbb{R}$. Then, f is continuous at any* **non***-cluster point $a \in D$.*

Proof: If a is **not** a cluster point, we can determine what this means by negating Definition 6.2.3: there does not exist a sequence (x_n) in $D \setminus \{a\}$ such that $x_n \to a$. We begin by proving that any sequence in D that converges to a is eventually constant (it has a *tail* consisting of terms all equal to a). This is equivalent to saying that any sequence (x_n) in D such that $x_n \to a$ contains finitely-many terms **not** equal to a. Assume to the contrary there are infinitely-many terms **not** equal to a. These terms form a subsequence (y_k) of (x_n) and, since they are not equal to a, this subsequence lives in $D \setminus \{a\}$. But by Theorem 3.1.3, we know $y_k \to a$, contradicting the fact a is **not** a cluster point. Hence, (x_n) is eventually constant, i.e. there exists $N \in \mathbb{N}$ such that $x_n = a$ for all $n \geq N$. Consequently, for any $\varepsilon > 0$, we have $|f(x_n) - f(a)| = 0 < \varepsilon$ for all $n \geq N$. In other words, $f(x_n) \to f(a)$ which means f is continuous at a. $\square$

Proposition 6.2.7 *Let $I \subseteq \mathbb{R}$ be an interval. Then, any $a \in I$ is a cluster point.*

Sketch of Proof: There are a number of cases depending on if I is open, closed or half-open. Let

$$x_n := \begin{cases} a + \dfrac{|\beta - a|}{2n} & \text{if} \quad \begin{array}{l} I = (\alpha, \beta), \text{ or} \\ I = [\alpha, \beta), \text{ or} \\ I = [\alpha, \beta] \textbf{ and } a = \alpha \end{array} \\ \\ a - \dfrac{|a - \alpha|}{2n} & \text{if} \quad \begin{array}{l} I = (\alpha, \beta], \text{ or} \\ I = [\alpha, \beta] \textbf{ and } a = \beta \end{array} \end{cases}.$$

In every case, the sequence (x_n) lives in $I \setminus \{a\}$ where $x_n \to a$ by the Algebra of Limits. □

6.3 Properties of Continuous Functions

Theorem 6.3.1 (Intermediate Value Theorem) *Let $f : [a, b] \to \mathbb{R}$ be continuous and $y \in \mathbb{R}$ be a number between $f(a)$ and $f(b)$. Then, there exists $c \in [a, b]$ such that $f(c) = y$.*

Proof: If $f(a) = f(b)$, the result is trivial as $y = f(a)$ is the only possibility; just choose $c = a$. We now assume $f(a) < f(b)$ without loss of generality (if the inequality is flipped, we repeat the following argument with the function $g := -f$). Define the set $S := \{x \in [a, b] : f(x) \leq y\}$. Certainly S is non-empty because $a \in S$ and it is bounded above by b. Therefore, the Axiom of Completeness implies that $c := \sup(S)$ exists. Because the interval $[a, b]$ is closed, we know that the supremum $c \in [a, b]$. Now, let (x_n) be a sequence in S such that $x_n \to c$ (this exists by Theorem 2.3.9). It follows that $f(x_n) \leq y$. By the continuity of f, we know that $f(x_n) \to f(c)$. As a consequence of the stability of closed inequalities under limits (Lemma 2.3.13), we know that $f(c) \leq y < f(b)$. Next, because c is the supremum of S, we know that $c + \frac{1}{n}$ is **not** in S. Because $c < b$, we know $c + \frac{1}{n} \in [a, b]$ for n sufficiently large. Therefore, for large enough n, we know that $f(c + \frac{1}{n}) > y$. Taking the limit then, $f(c + \frac{1}{n}) \to f(c) \geq y$. Hence, $f(c) = y$. □

Remark We can interpret the Intermediate Value Theorem geometrically in a rather easy way. Indeed, given a continuous function on an interval (or any function which is continuous when restricted to some interval), every value between the start and end values is obtained by some input inside the interval. The picture for this is shown in Figure 2 below.

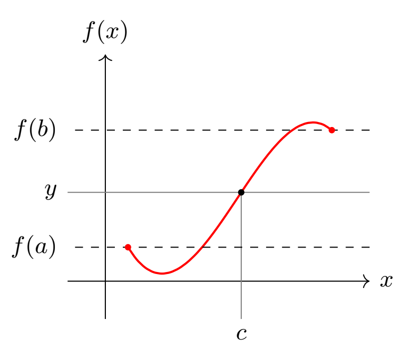

Figure 2: The geometric interpretation of the Intermediate Value Theorem.

Corollary 6.3.2 *Let $f : D \to \mathbb{R}$ be continuous and $[a, b] \subseteq D$ be contained in the domain. Then, f takes every value between $f(a)$ and $f(b)$ on D.*

Proof: Simply apply the Intermediate Value Theorem to the restriction of f to $[a, b]$. □

Lemma *Let $n \in \mathbb{N}$. Then, $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = x^n$ is strictly increasing.*

Proof: Let $y > x > 0$. Then, we can factorise the following:

$$f(y) - f(x) = y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}).$$

But the first factor $y - x > 0$ and the second factor $y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1} \geq y^{n-1} > 0$. This implies that $f(y) - f(x) > 0$, meaning precisely that f is strictly increasing. □

Theorem 6.3.3 (Existence of Roots) *Let $n \in \mathbb{N}$. Given any $y \geq 0$, there exists a unique $x \geq 0$ such that $x^n = y$. We denote this real number by $y^{1/n}$ and call it the n^{th} root of y.*

Proof: For **existence**, let $f : [0, y + 1] \to \mathbb{R}$ be given by $f(x) = x^n - y$. This is a polynomial and thus is continuous (Corollary 6.1.4). Notice $f(0) = -y \leq 0$. Also, the previous lemma implies

$$f(y + 1) = (y + 1)^n - y \geq y + 1 - y = 1 > 0.$$

Because $f(0)$ is negative and $f(y+1)$ is positive, we know 0 is a real number between them. We can now apply the Intermediate Value Theorem (or Corollary 6.3.2): there exists $x \in [0, y + 1]$ such that $f(x) = 0$, that is $x^n = y$. For **uniqueness**, let $z \in [0, \infty)$ also satisfy $f(z) = 0$. Then, $f(z) = f(x)$. But the previous lemma says f is *strictly* increasing, so $z < x$ and $z > x$ would each be a contradiction to this result. Consequently, the only option remaining is $z = x$. □

Note: For each $n \in \mathbb{N}$, we have a well-defined (strictly increasing and continuous) function

$$\sqrt[n]{\cdot} : [0, \infty) \to \mathbb{R}, \qquad x \mapsto x^{1/n}.$$

Definition 6.3.4 Let $f : D \to \mathbb{R}$ be a function.

(i) We say f is bounded above if there exists $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in D$.
(ii) We say f is bounded below if there exists $L \in \mathbb{R}$ such that $f(x) \geq L$ for all $x \in D$.
(iii) We say f is bounded if it is both bounded above and below.
(iv) We say f attains a minimum if there exists $c \in D$ with f bounded below by $f(c)$.
(v) We say f attains a maximum if there exists $c \in D$ with f bounded above by $f(c)$.

Remark A function may be bounded but **not** attain a maximum value. It will, however, still have a supremum (guaranteed by the Axiom of Completeness). For example, the function $f(x) = -\frac{1}{x}$ is bounded above by 0, but it never attains this bound. Conversely, **if** a function attains a maximum value, it is automatically bounded above and its maximum value is its supremum.

Note: Boundedness of f is just boundedness of the **set** $f(D)$ as in Definition 1.1.7.

Theorem 6.3.5 (Extreme Value Theorem) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, f is both bounded above and below, and attains a maximum and minimum.*

Proof: Assume to the contrary that f is unbounded above. Then, each $n \in \mathbb{N}$ is **not** an upper bound on f, so there exists a number $x_n \in [a, b]$ such that $f(x_n) > n$. This defines a sequence (x_n) which is bounded; the Bolzano-Weierstrass Theorem implies the existence of a convergent subsequence, say $x_{n_k} \to x$. Because $a \leq x_{n_k} \leq b$ for all $k \in \mathbb{N}$, we know the limit $a \leq x \leq b$ (Lemma 2.3.13). Since f is continuous, we know that $f(x_{n_k}) \to f(x)$. Because convergent sequences are bounded (Theorem 2.3.4), we know that the sequence $(f(x_{n_k}))$ is bounded. But we assumed $f(x_{n_k}) > n_k \geq k$, so the sequence $(f(x_{n_k}))$ is unbounded above, a contradiction.

The image $f([a, b]) \coloneqq \{f(t) : t \in [a, b]\} \subseteq \mathbb{R}$ of this function is non-empty and bounded above, so the Axiom of Completeness guarantees that $M \coloneqq \sup(f)$ exists. For any $n \in \mathbb{N}$, we see that $M - \frac{1}{n} < M$, meaning $M - \frac{1}{n}$ is **not** an upper bound on f. Hence, there exists $y_n \in [a, b]$ whose image $f(y_n) \geq M - \frac{1}{n}$. Combining this with the definition of the supremum, we conclude that

$$M - \frac{1}{n} \leq f(y_n) \leq M.$$

Since (y_n) is bounded, it has a convergent subsequence by the Bolzano-Weierstrass Theorem (this is the same argument used in the contradiction done above), say (y_{n_k}) such that $y_{n_k} \to c$. By the Squeeze Theorem applied to the inequalities above, we see that

$$M \leq f(c) \leq M,$$

where we have used the continuity of f to conclude $f(y_{n_k}) \to f(c)$. Thus, $f(c) = M = \sup(f)$. To see that f is bounded below and attains a minimum value, a near-identical argument can be done. Alternatively, one can apply the above to the function $g : [a, b] \to \mathbb{R}$ where $g(x) \coloneqq -f(x)$. □

Corollary 6.3.6 *Continuous functions send closed intervals to closed intervals.*

Proof: Let $f : D \to \mathbb{R}$ and $[a, b] \subseteq D$. We can apply the Extreme Value Theorem to the restriction of f to $[a, b]$. Indeed, there exist $c, d \in [a, b]$ such that $f(c) = M$ is the maximum and $f(d) = L$ is the minimum of f on $[a, b]$. In other words, $f(x) \in [L, M]$ for all $x \in D$. This establishes that $f([a, b]) \subseteq [L, M]$. On the other hand, we apply the Intermediate Value Theorem to the restriction of f to $[\min\{c, d\}, \max\{c, d\}]$. Indeed, for each $y \in [L, M]$, there exists $x \in [\min\{c, d\}, \max\{c, d\}]$ such that $f(x) = y$. This establishes that $[L, M] \subseteq f([a, b])$. Combining both inclusions tells us $f([a, b]) = [L, M]$, as required. □

Note: This corollary says that any number between two outputs $y, z \in f([a, b])$ is also an output of the function f, even if y and z are **not** between $f(a)$ and $f(b)$. Consequently, one can think of this as a sort-of strengthening of the Intermediate Value Theorem.

Corollary 6.3.7 *Continuous functions send intervals to intervals.*

Sketch of Proof: Let $f : D \to \mathbb{R}$ and $I \subseteq D$ an interval. Take $a \in I$, so that we can refer to an element $f(a) \in f(I)$ in the image. There are a number of possibilities depending on the interval.

- Let f be bounded above **and** attain its maximum $f(c) = M$. Necessarily then, $f(a) \leq M$ by definition of a maximum. Hence, f takes every value between $f(a)$ and M by the Intermediate Value Theorem. The image contains every value in $[f(a), M]$.
- Let f be bounded above but **not** attain its maximum. It still has a supremum $M := \sup(f)$. Necessarily then, $f(x) < M$ for all $x \in I$ and, for any $y \in \mathbb{R}$ with $f(a) \leq y < M$, there exists $x \in I$ such that $f(x) > y$. Hence, f takes every value between $f(a)$ and $f(x)$ by the Intermediate Value Theorem. The image contains every value in $[f(a), M)$.
- Let f be **unbounded** above. In this case, for any $z \in \mathbb{R}$ with $f(a) \leq z$, there exists $x \in I$ such that $z < f(x)$. Again, f takes every value between $f(a)$ and $f(x)$ by the Intermediate Value Theorem. The image contains every value in $[f(a), \infty)$.
- Let f be bounded below **and** attain its minimum $f(c) = L$. Necessarily then, $L \geq f(a)$ by definition of a minimum. Hence, f takes every value between L and $f(a)$ by the Intermediate Value Theorem. The image contains every value in $[L, f(a)]$.
- Let f be bounded below but **not** attain its minimum. It still has an infimum $L := \inf(f)$. Necessarily then, $L < f(x)$ for all $x \in I$ and, for any $y \in \mathbb{R}$ with $L < y \leq f(a)$, there exists $x \in I$ such that $f(x) < y$. Hence, f takes every value between $f(x)$ and $f(a)$ by the Intermediate Value Theorem. The image contains every value in $(L, f(a)]$.
- Let f be **unbounded** below. In this case, for any $z \in \mathbb{R}$ with $z \leq f(a)$, there exists $x \in I$ such that $f(x) < z$. Again, f takes every value between $f(x)$ and $f(a)$ by the Intermediate Value Theorem. The image contains every value in $(-\infty, f(a)]$.

The image $f(I)$ is then the interval whose endpoints are determined by what we discussed above, e.g. if the interval had the form $I = (-\infty, b]$, then the image has the form $f(I) = (-\infty, M]$. $\square$

6.4 Limits at Infinity

Definition 6.4.1 A real sequence (a_n) diverges (or tends) to infinity if, for each $K \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $a_n > K$. In this case, we write $a_n \to \infty$.

Remark We explain Definition 6.4.1 in words and with a geometric interpretation in Figure 3. Given a sequence (a_n), we can show that it diverges to infinity by showing that for **any** number ($K \in \mathbb{R}$), there exists a point in the sequence a_N (there exists $N \in \mathbb{N}$) after which (for all $n \geq N$) it and every subsequent term in the sequence exceeds that number ($a_n > K$). Geometrically, if we plot n against a_n on a pair of axes, then after N, every point will live **above** the line $a_n = K$.

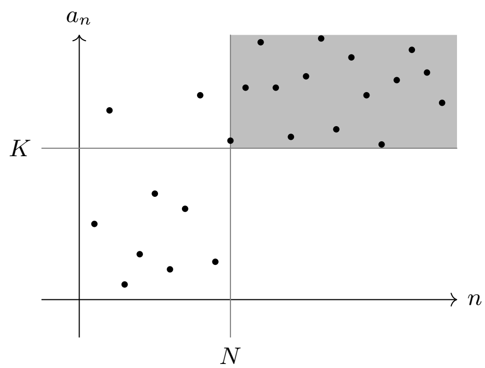

Figure 3: The geometric interpretation of the divergence of (a_n) to infinity.

Definition 6.4.2 A real sequence (a_n) diverges (or tends) to minus infinity if, for each $K \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $a_n < K$. In this case, we write $a_n \to -\infty$.

Note: In fact, Definitions 6.4.1 and 6.4.2 are related: $a_n \to -\infty$ if and only if $-a_n \to \infty$.

Definition 6.4.3 Let $f : D \to \mathbb{R}$ be a function

(i) If $D \subseteq \mathbb{R}$ is unbounded **above**, we say f has a limit $L \in \mathbb{R}$ at infinity if, for any sequence (x_n) in D with $x_n \to \infty$, we have $f(x_n) \to L$. We write $\lim_{x \to \infty} f(x) = L$.

(ii) If D is unbounded **below**, we say f has a limit $K \in \mathbb{R}$ at minus infinity if, for any sequence (x_n) in D with $x_n \to -\infty$, we have $f(x_n) \to K$. We write $\lim_{x \to -\infty} f(x) = K$.

Theorem 6.4.4 *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and both $\lim_{x \to \pm\infty} f(x)$ exist, then f is bounded.*

Proof: Let $L := \lim_{x \to \infty} f(x)$ and $K := \lim_{x \to -\infty} f(x)$. First, there exists $X \in \mathbb{R}$ where, for all $x > X$,

$$|f(x) - L| < 1 \qquad \Leftrightarrow \qquad L - 1 < f(x) < L + 1.$$

Indeed, suppose to the contrary this is not the case. In particular, it is true that $|f(x) - L| \geq 1$ for each $x \in \mathbb{R}$ with $x > n$, for each $n \in \mathbb{N}$. Choosing some $x_n > n$, this defines for us a sequence (x_n) whereby $|f(x_n) - L| \geq 1$. Notice that (x_n) diverges to infinity and $(f(x_n))$ does **not** converge to L. This contradicts the fact that f has a limit L at infinity. Similarly, we can conclude there exists $Y \in \mathbb{R}$ where, for all $x > Y$,

$$|f(x) - K| < 1 \qquad \Leftrightarrow \qquad K - 1 < f(x) < K + 1.$$

If $Y > X$, we are done. Otherwise, we can apply the Extreme Value Theorem to the restriction of f on $[Y, X]$. Since f is bounded on this interval, it is certainly bounded on all of $\mathbb{R}$. $\square$

7 Complex Sequences and Series

Reminder: A complex number $z \in \mathbb{C}$ has the form $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. We call $x =: \operatorname{Re}(z)$ the real part of z and $y =: \operatorname{Im}(z)$ the imaginary part of z. The modulus is the real number $|z| := \sqrt{x^2 + y^2}$. The conjugate is the complex number $\bar{z} := x - iy$.

Note: We can prove by direct computation that $|z|^2 = z\bar{z}$, $|\bar{z}| = |z|$ and $\overline{z+w} = \bar{z} + \overline{w}$.

7.1 Convergence of Complex Sequences

Proposition 7.1.1 *Let $z, w \in \mathbb{C}$. Then, the following are true:*

(i) $\operatorname{Re}(z) \leq |z|$.

(ii) $\operatorname{Im}(z) \leq |z|$.

(iii) $|zw| = |z||w|$. (**Multiplicativity**)

(iv) $|z + w| \leq |z| + |w|$. (**Triangle Inequality**)

Proof: (i) Squaring both sides, we want to show $x^2 \leq x^2 + y^2$. This is always true as $y^2 \geq 0$.

(ii) Again, squaring both sides, we want to show $y^2 \leq x^2 + y^2$. This is always true as $x^2 \geq 0$.

(iii) Let $z = x + iy$ and $w = a + ib$, so $zw = (x + iy)(a + ib) = (xa - yb) + (xb + ya)i$. Working with the squares of the moduli (and taking the square root after), we obtain what we want:

$$\begin{aligned}
|zw|^2 &= (xa - yb)^2 + (xb + ya)^2 \\
&= (x^2a^2 - 2xyab + y^2b^2) + (x^2b^2 + 2xyab + y^2a^2) \\
&= x^2a^2 + y^2b^2 + x^2b^2 + y^2a^2 \\
&= (x^2 + y^2)(a^2 + b^2) \\
&= |z|^2|w|^2.
\end{aligned}$$

(iv) Assume to the contrary that $|z + w| > |z| + |w|$. Squaring both sides, we obtain

$$\begin{aligned}
& & |z+w|^2 &> (|z| + |w|)^2 \\
&\Rightarrow & (z+w)(\overline{z+w}) &> |z|^2 + 2|z||w| + |w|^2 \\
&\Leftrightarrow & (z+w)(\bar{z} + \overline{w}) &> |z|^2 + 2|z||w| + |w|^2 \\
&\Leftrightarrow & |z|^2 + z\overline{w} + w\bar{z} + |w|^2 &> |z|^2 + 2|z||w| + |w|^2 \\
&\Rightarrow & z\overline{w} + w\bar{z} &> 2|z||w| \\
&\Leftrightarrow & z\overline{w} + w\bar{z} &> 2|z||\overline{w}| \\
&\Leftrightarrow & 2\operatorname{Re}(z\overline{w}) &> 2|z\overline{w}|,
\end{aligned}$$

using part(iii), but this is a contradiction to the inequality established in part (i). $\square$

Definition A sequence of complex numbers is a function $z : \mathbb{N} \to \mathbb{C}$ with output denoted z_n (instead of the usual notation $z(n)$ for functions). A term in the sequence is denoted by z_n, whereas the whole sequence is denoted by $(z_n)_{n\in\mathbb{N}}$, or just (z_n) for short.

Think of a complex sequence as an infinite ordered list $(z_n) = (z_1, z_2, z_3, \dots)$ of complex numbers.

Definition 7.1.2 A complex sequence (z_n) converges to the number $L \in \mathbb{C}$ if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $|a_n - L| < \varepsilon$. Here, we write $z_n \to L$.

Remark Comparing this with Definition 2.2.3 (the convergence of *real* sequences), we see it is very much the same idea: for any positive number, there is a point in the sequence after which it and all subsequent terms lie within distance that positive number of L. The main difference comes from the geometric interpretation: if we plot the outputs z_n in the complex plane, every point for $n \geq N$ will live **inside** the (open) disk of radius ε centred at the complex number L.

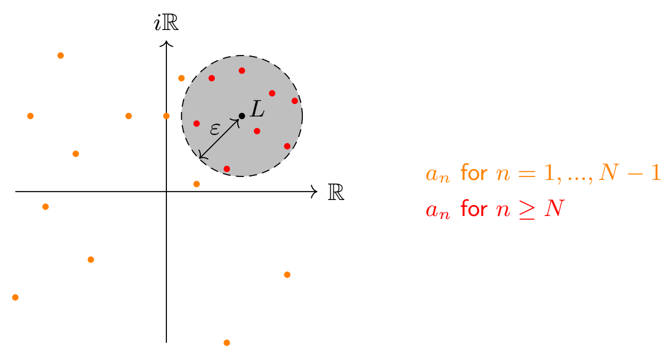

Figure 4: The geometric interpretation of the convergence of a complex sequence (z_n).

Proposition 7.1.4 *Let (z_n) be a complex sequence and $L \in \mathbb{C}$, with each $z_n = x_n + iy_n$ for $x_n, y_n \in \mathbb{R}$ and $L = A + iB$ for $A, B \in \mathbb{R}$. Then, the following are equivalent:*

(i) $z_n \to L$.

(ii) $x_n \to A$ *and* $y_n \to B$ *in the sense of* Definition 2.2.3.

(iii) $|z_n - L| \to 0$.

Proof: ((i) ⇒ (ii)) Let $\varepsilon > 0$ be given. As $z_n \to L$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|z_n - L| < \varepsilon$. If we continue to assume that $n \geq N$, we obtain the following as a consequence:

$$\begin{aligned} |x_n - A| &= \sqrt{(x_n - A)^2} \\ &\leq \sqrt{(x_n - A)^2 + (y_n - B)^2} \\ &= |(x_n - A) + i(y_n - B)| \\ &= |(x_n + iy_n) - (A + iB)| \\ &= |z_n - L| \\ &< \varepsilon. \end{aligned}$$

This establishes $x_n \to A$, and an identical argument can be repeated to show that $y_n \to B$ too.

((ii) ⇒ (iii)) Let $\varepsilon > 0$ be given. Since $x_n \to A$, there exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|x_n - A| < \frac{\varepsilon}{2}$. Similarly, as $y_n \to B$, there exists $N_2 \in \mathbb{N}$ such that, for all $n \geq N_2$, $|y_n - B| < \frac{\varepsilon}{2}$. For $n \geq \max\{N_1, N_2\}$, we can use the Triangle Inequality from Proposition 7.1.1 to see that

$$\begin{aligned}||z_n - L| - 0| &= |z_n - L| \\ &= |(x_n - A) + i(y_n - B)| \\ &\leq |x_n - A| + |y_n - B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon,\end{aligned}$$

which is precisely to say that $|z_n - L| \to 0$, as we required.

((iii) ⇒ (i)) This is essentially completed above. Indeed, for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $||z_n - L| - 0| < \varepsilon$. But this is the same as $|z_n - L| < \varepsilon$. ☐

Corollary (Uniqueness of Limits) *The limit of a convergent complex sequence is unique.*

Proof: Let $z_n \to L$. By writing $z_n = x_n + iy_n$, it follows from Proposition 7.1.4 that the real sequences (x_n) and (y_n) converge, and their limits are unique by Theorem 2.2.6. Explicitly, $x_n \to A$ and $y_n \to B$ for unique $A, B \in \mathbb{R}$. But this uniquely defines the limit $L = A + iB$. ☐

Corollary *Any convergent complex sequence is bounded.*

Proof: Let $z_n \to L$. By writing $z_n = x_n + iy_n$, it follows from Proposition 7.1.4 that the real sequences (x_n) and (y_n) converge, and thus they are bounded by Theorem 2.3.4. Explicitly, there exist $K, M > 0$ such that $|x_n| \leq K$ and $|y_n| \leq M$ for all n. By the Triangle Inequality, we can conclude that $|z_n| = |x_n + iy_n| \leq |x_n| + |y_n| \leq K + M$. In other words, (z_n) is bounded. ☐

Corollary (Algebra of Limits) *Let $z_n \to Z$ and $w_n \to W$. Then, the following are true:*
(i) $z_n + w_n \to Z + W$.
(ii) $\lambda z_n \to \lambda Z$ *for all* $\lambda \in \mathbb{C}$.
(iii) $z_n w_n \to ZW$.
(iv) $z_n / w_n \to Z/W$ **if** $w_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $W \neq 0$.

Sketch of Proof: We can write each term in (z_n) and (w_n) in terms of real numbers and apply the usual Algebra of Limits in conjunction with Proposition 7.1.4 to conclude the result. ☐

Note: Generally, the results we proved about convergent *real* sequences pass over the convergent *complex* sequences by viewing things through the lens of Proposition 7.1.4.

Definition 7.1.8 Let $D \subseteq \mathbb{C}$. Then, $f : D \to \mathbb{C}$ is continuous at $w \in D$ if, for **all** sequences (z_n) in D where $z_n \to w$, we have $f(z_n) \to f(w)$. If f is **not** continuous at $w \in D$, we say it is discontinuous at $w \in D$. We say f is continuous if it is continuous at **every** $w \in D$.

Proposition 7.1.9 *Let $f : D \to \mathbb{C}$ be continuous and (z_n) an inductively-defined sequence given by iterating f from an initial value $z_0 \in \mathbb{C}$, meaning $z_n = f(z_{n-1})$. If $z_n \to L$, then L is a* fixed point *of f, meaning $f(L) = L$.*

Proof: Let $z_n \to L$. As noted above, we can carry over results like Theorem 3.1.3 (subsequences of convergent sequences themselves converge) to work for *complex* sequences. As such, it follows that the subsequence $z_{n+1} \to L$ also. By the inductive definition of (z_n), we have $z_{n+1} = f(z_n)$. The continuity of f guarantees $f(z_n) \to f(L)$, and the Uniqueness of Limits gives $f(L) = L$. $\square$

7.2 Complex Series

Definition A complex series is a sequence of complex numbers (s_k) with terms defined by

$$s_k = \sum_{n=1}^{k} z_n,$$

where z_n is the n^{th} summand and s_k is the k^{th} partial sum. The series is denoted $\sum_{n=1}^{\infty} z_n$.

This is very much the same as Definition 4.1.1, except we have replaced "real" by "complex".

Note: $\sum_{n=1}^{\infty} z_n$ is convergent if the sequence of partial sums (s_k) converges in the usual sense.

Remark One can write the complex series in terms of two real series in the expected way:

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (x_n + iy_n) = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n.$$

Proposition 7.1.4 implies the series converges if and only if the corresponding **real** series converge.

Definition 7.2.1 The series $\sum_{n=1}^{\infty} z_n$ converges absolutely if the **real** series $\sum_{n=1}^{\infty} |z_n|$ converges.

Proposition 7.2.3 *If the real series $\sum_{n=1}^{\infty} |z_n|$ converges, then $\sum_{n=1}^{\infty} z_n$ converges.*

Proof: Let $z_n = x_n + iy_n$ and (t_k) be the sequence of partial sums of $\sum_{n=1}^{\infty} |z_n|$. Let's also define

$$u_k := \sum_{n=1}^{k} |x_n| \qquad \text{and} \qquad v_k := \sum_{n=1}^{k} |y_n|.$$

Since we assume absolute convergence, it means the real sequence (t_k) converges in the usual sense. Hence, it is bounded above (because the sequence of partial sums is increasing). From Proposition 7.1.1(i) and (ii), we have $x_n \leq |z_n|$ and $y_n \leq |z_n|$, respectively. Hence, it follows that

$$u_k \leq t_k \qquad \text{and} \qquad v_k \leq t_k.$$

Since (t_k) is bounded above, so too are the real sequences (u_k) and (v_k). Moreover, each of these is also increasing; the Monotone Convergence Theorem implies that they converge. Because the absolute convergence of **real** series implies convergence (Theorem 4.3.1), we know these converge:

$$\sum_{n=1}^{\infty} x_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} y_n.$$

Finally, the previous remark now implies that $\sum_{n=1}^{\infty} z_n$ converges, precisely what we wanted. □

7.3 Power Series

Definition Let $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$. A (complex) power series is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where z is a complex variable. The terms in the series are $a_n z^n$ and the k^{th} partial sum is

$$s_k(z) = \sum_{n=0}^{k} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k.$$

Notation Up until this point, we have used z_n to denote *complex* terms as opposed to a_n which we used for *real* terms. However, the above definition assumes that $a_n \in \mathbb{C}$. The reason for the notation switch is so that the power series terms $a_n z^n$ look appealing to the eye; if we insisted on still using z_n, our terms would be $z_n z^n$ which is a little confusing at a glance!

Note: Our power series begin at $n = 0$, which is different to the $n = 1$ start as before.

Theorem (Complex Geometric Series) *The series* $\sum_{n=0}^{\infty} z^n$ *converges if and only if* $|z| < 1$.

Proof: As in the proof of the convergence of the *real* geometric series, the k^{th} partial sum s_k is

$$\begin{aligned} s_k &= 1 + z + z^2 + \cdots + z^k \\ \Rightarrow \qquad z s_k &= z + z^2 + z^3 + \cdots + z^{k+1} \\ \Rightarrow \qquad (1-z)s_k &= 1 - z^{k+1} \\ \Rightarrow \qquad s_k &= \frac{1-z^{k+1}}{1-z}. \end{aligned}$$

($\Leftarrow$) If $|z| < 1$, then $|z|^{k+1} \to 0$ and so $s_k \to \frac{1}{1-z}$ by the Algebra of Limits; we have convergence.

($\Rightarrow$) If $|z| \geq 1$, then $|z^n| \geq 1$. Since they are complex numbers, we can write each $z^n = x_n + iy_n$. Thus, the previous inequality is equivalent to $x_n^2 + y_n^2 \geq 1$. By Lemma 2.3.13, we **cannot** have **both** $x_n \to 0$ and $y_n \to 0$ (if so, this would imply the obviously-false inequality $0^2 + 0^2 \geq 1$). It must then be true that $x_n \not\to 0$ or $y_n \not\to 0$. It now follows from the Divergence Test that

$$\sum_{n=1}^{\infty} x_n \text{ diverges} \qquad \text{or} \qquad \sum_{n=1}^{\infty} y_n \text{ diverges.}$$

Therefore, the previous remark implies the complex geometric series diverges for all $|z| \geq 1$. □

Note: Generally, a power series may converge for some values of z and not for others. As such, it defines a complex function on a subset of $\mathbb{C}$. Note however that every power series converges when $z = 0$. Indeed, every partial sum $s_k(0) = a_0 \to a_0$ in this situation.

Definition 7.3.2 The exponential, sine and cosine functions are defined as follows:

$$\begin{aligned} \exp : \mathbb{C} \to \mathbb{C}, \qquad & \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\ \sin : \mathbb{C} \to \mathbb{C}, \qquad & \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \\ \cos : \mathbb{C} \to \mathbb{C}, \qquad & \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}. \end{aligned}$$

Proposition *The function* exp *is well-defined, that is* $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ *converges for all* $z \in \mathbb{C}$.

Proof: The above note means we need only prove it for $z \neq 0$. For $w_n := \frac{z^n}{n!} > 0$, notice that

$$\frac{|w_{n+1}|}{|w_n|} = \frac{|z|^{n+1}}{(n+1)!} \frac{n!}{|z|^n} = \frac{|z|}{n+1} \to 0 < 1.$$

By the Ratio Test, $\sum_{n=0}^{\infty} |w_n|$ converges. Thus, $\exp(z) = \sum_{n=0}^{\infty} w_n$ converges by Proposition 7.2.3. □

Definition 7.3.5 The radius of convergence of a power series $\sum_{n=0}^{\infty} a_n z^n$ is the real constant

$$R := \sup\{|z| : \sum_{n=0}^{\infty} |a_n z^n| \text{ converges}\}.$$

If the set on the right-hand side is unbounded, there is no supremum and we say "$R = \infty$".

In other words, the radius of convergence is the positive constant $R \geq 0$ such that the power series converges for all values of $z \in \mathbb{C}$ within that distance from the origin (that is $|z| < R$) and diverges for all values more than that distance away from the origin (that is $|z| > R$).

Note: We can't immediately tell if the series converges or diverges at $z \in \mathbb{C}$ with $|z| = R$.

Lemma 7.3.6 *If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = w$, then it converges absolutely for $|z| < |w|$.*

Proof: We assume $\sum_{n=0}^{\infty} a_n w^n$ converges; writing $a_n w^n = x_n + iy_n$, the previous remark implies

$$\sum_{n=0}^{\infty} x_n \text{ converges} \qquad \text{and} \qquad \sum_{n=0}^{\infty} y_n \text{ converges.}$$

By the Divergence Test, it follows that the real sequences $x_n \to 0$ and $y_n \to 0$. Therefore, the sequence of terms $a_n w^n \to 0 + i0 = 0$ by Proposition 7.1.4. In particular, this sequence is bounded, meaning there exists $K > 0$ such that $|a_n w^n| < K$ for all n. Consequently, notice that

$$|a_n z^n| = |a_n w^n| \frac{|z^n|}{|w^n|} < K \frac{|z^n|}{|w^n|}.$$

Hence, by the Comparison Test with $\sum_{n=0}^{\infty} \frac{|z^n|}{|w^n|}$, it follows that $\sum_{n=0}^{\infty} |a_n z^n|$ converges, as required. □

Theorem 7.3.7 *Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. Then,*

the series converges absolutely for $|z| < R$ **and** *the series diverges for $|z| > R$.*

Proof: Consider an arbitrary power series $\sum_{n=0}^{\infty} a_n z^n$, and let us introduce the notation

$$A := \{|z| : \sum_{n=0}^{\infty} |a_n z^n| \text{ converges}\} \subseteq \mathbb{R}$$

so that $R = \sup(A)$. If $|z| < R$, there exists $w \in \mathbb{C}$ with $|z| < |w| < R$ such that this converges:

$$\sum_{n=0}^{\infty} |a_n w^n|.$$

We know such w exists; if not, R would **not** be the the **least** upper bound on A. But complex absolute convergence implies convergence (Proposition 7.2.3), and we can apply Lemma 7.3.6 to conclude that the original arbitrary power series converges. On the other hand, if $|z| > R$, assume to the contrary that the power series converges. Again by Lemma 7.3.6, it follows that

$$\sum_{n=0}^{\infty} |a_n v^n| \text{ converges,} \qquad \text{for } v = \frac{1}{2}(|z| + R).$$

This is because $|v| < |z|$. However, $|v| \in A$ and yet $|v| > R$ by the assumption $|z| > R$. In other words, R is **not** an upper bound on A, contradicting the fact it is the radius of convergence. □

Corollary *A power series* $\sum_{n=0}^{\infty} a_n z^n$ *defines a complex function* $f : \{z \in \mathbb{C} : |z| < R\} \to \mathbb{C}$.

Proof: This is simply another way to re-phrase Theorem 7.3.7. □

Method – Finding the Radius of Convergence: Consider the power series $\sum_{n=0}^{\infty} a_n z^n$.

(i) Let $b_n := a_n z^n$ be the sequence of terms in the series.
(ii) Compute the ratio $\frac{|b_{n+1}|}{|b_n|}$ and determine its limit L as $n \to \infty$ in terms of $|z|$.
(iii) Apply the Ratio Test to find an upper bound on $|z|$ that makes the limit $L < 1$.

Note: Although the above method uses the Ratio Test to find the radius of convergence, notice that the **definition** of the radius of convergence has **no** mention of the Ratio Test.

Remark Since $\mathbb{R} \subseteq \mathbb{C}$, we can equally apply all of the above theory to the case of a *real* power series. This is defined in precisely the same way, except the coefficients $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.