MATH1026 Sets, Sequences and Series

Cheatsheet

2021/22

This document collects together the important definitions and results presented throughout the lecture notes. The numbering of the sections will be consistent with that in the lecture notes.

Contents

1 The Set of Real Numbers

1.1 Axiomatic Characterisation of the Real Numbers

Definition 1.1.1 A set $\mathbb{K}$ with two binary operations $+ : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ and $\cdot : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ and a relation $\leq$ is an ordered field if, for all $x, y, z \in \mathbb{K}$, the following hold true:

(A1) $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. (**Associativity**)
(A2) $x + y = y + x$ and $x \cdot y = y \cdot x$. (**Commutativity**)
(A3) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$. (**Distributivity**)
(A4) There exists a unique element $0 \in \mathbb{K}$ with $0 + x = x$. (**Additive Identity**)
(A5) There exists a unique element $1 \in \mathbb{K}$ with $1 \cdot x = x$. (**Multiplicative Identity**)
(A6) For each $x \in \mathbb{K}$, there is a unique $y \in \mathbb{K}$ with $x + y = 0$. (**Additive Inverse**)
(A7) For each $x \in \mathbb{K}$, there is a unique $y \in \mathbb{K}$ with $x \cdot y = 1$. (**Multiplicative Inverse**)
(A8) $x \leq y$ implies $x + z \leq y + z$.
(A9) $x \leq y$ and $y \leq z$ implies $x \leq z$. (**Transitivity**)
(A10) $x \leq y$ and $y \leq x$ implies $x = y$. (**Anti-Symmetry**)
(A11) $x \leq y$ and $0 \leq z$ implies $x \cdot z \leq y \cdot z$.
(A12) Either $x \leq y$ or $y \leq x$. (**Strong Connectivity**)

Note: We often relabel the additive/multiplicative inverses in (A6) and (A7) as follows:
(i) The additive inverse of x is denoted $-x$.
(ii) The multiplicative inverse of x is denoted x^{-1} or $\frac{1}{x}$.

Theorem 1.1.3 *For any $a, b, c \in \mathbb{K}$ elements of an ordered field, we have the following:*
(i) $a + c = b + c$ *implies* $a = b$.
(ii) $a \cdot 0 = 0$.
(iii) $(-a) \cdot b = -(a \cdot b)$.
(iv) $(-a) \cdot (-b) = a \cdot b$.
(v) $a \cdot c = b \cdot c$ *with* $c \neq 0$ *implies* $a = b$.
(vi) $a \cdot b = 0$ *implies* $a = 0$ *or* $b = 0$.

Theorem 1.1.4 *For any $a, b, c \in \mathbb{K}$ elements of an ordered field, we have the following:*
(i) $a \leq b$ *implies* $-b \leq -a$.
(ii) $a \leq b$ *and* $c \leq 0$ *implies* $b \cdot c \leq a \cdot c$.
(iii) $a \leq 0$ *and* $b \leq 0$ *implies* $0 \leq a \cdot b$.
(iv) $0 \leq a \cdot a =: a^2$.
(v) $0 < 1$.
(vi) $0 < a$ *implies* $0 < a^{-1}$.
(vii) $0 < a < b$ *implies* $0 < b^{-1} < a^{-1}$.

Although you should take care to remember these results, they should be very familiar; this is the sort of thing we have been working with since high school (or even before) except now we are interested in **any** ordered field, not just $\mathbb{R}$. We haven't proved that $\mathbb{R}$ is an ordered field though!

Definition 1.1.6 Let $\mathbb{K}$ be an ordered field. A subset $S \subseteq \mathbb{K}$ is called bounded above if there exists an element $M \in \mathbb{K}$ such that $s \leq M$ for all $s \in S$. We then call M an upper bound for S. Similarly, a subset $S \subseteq \mathbb{K}$ is called bounded below if there exists an element $L \in \mathbb{K}$ such that $s \geq L$ for all $s \in S$. We then call L a lower bound for S.

Note: For a subset $S \subseteq \mathbb{K}$, we define $-S \coloneqq \{-s : s \in S\} \subseteq \mathbb{K}$. Then, S is bounded above if and only if $-S$ is bounded below (and vice versa); this is a useful trick for some proofs.

Definition 1.1.9 Let $\mathbb{K}$ be an ordered field and $S \subseteq \mathbb{K}$ a subset. We call M the least upper bound for S if it is an upper bound for S and there is **no** upper bound smaller than M.

Note: If $|S| < \infty$ (S is finite), a least upper bound **always** exists; the largest element of S.

Definition 1.1.11 An ordered field $\mathbb{K}$ is complete if **every** non-empty subset that is bounded from above has a least upper bound.

Axiom (Axiom of Completeness) *$\mathbb{R}$ with the usual operations is a complete ordered field.*

Remark 1.1.12 We can restate the Axiom of Completeness in terms of lower bounds and greatest lower bounds; this is perfectly valid. In this way, a greatest lower bound is just a lower bound such that there is **no** lower bound bigger than it.

1.2 The Structure of the Set of Real Numbers

Definition 1.2.1 Let $S \subseteq \mathbb{R}$ be a non-empty subset. The supremum of S is the least upper bound of S, denoted $\sup(S)$, **if** the set is bounded above. The infimum of S is the greatest lower bound of S, denoted $\inf(S)$, **if** the set is bounded below.

Note: If S isn't bounded from above/below, we would write $\sup(S) = \infty$ or $\inf(S) = -\infty$, respectively. Moreover, we could also define the infimum as $\inf(S) \coloneqq -\sup(-S)$.

Theorem 1.2.3 (Archimedean Property of $\mathbb{R}$) *For all $x \in \mathbb{R}$, there exists $N \in \mathbb{N}$ with $x < N$.*

Proof: Assume to the contrary this is not the case, meaning $\mathbb{N}$ is bounded above by x. Then, by the Axiom of Completeness, $\mathbb{N}$ has a least upper bound $y \in \mathbb{R}$. By definition, $y - 1$ is **not** an upper bound on $\mathbb{N}$, which means there is an element of the naturals larger than it, i.e. there exists $M \in \mathbb{N}$ with $M \geq y - 1$. This is equivalent to $y \leq M + 1$, but $M + 1 \in \mathbb{M}$, so y is **not** an upper bound on $\mathbb{N}$, a contradiction to it being the least upper bound. $\square$

Corollary 1.2.4 *For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $0 < \frac{1}{N} < \varepsilon$.*

Proof: By the Archimedean Property of $\mathbb{R}$, we can choose $N \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < N$. □

Corollary 1.2.5 *For all $\delta > 0$ and $x > 0$, there exists $N \in \mathbb{N}$ with $N\delta > x$.*

Proof: By the Archimedean Property of $\mathbb{R}$, we can choose $N \in \mathbb{N}$ such that $\frac{x}{\delta} < N$. □

Theorem 1.2.6 (Density of $\mathbb{Q}$ in $\mathbb{R}$) *Let $x \in \mathbb{R}$. Then, for each $\varepsilon > 0$, there exists a rational $q \in \mathbb{Q}$ where $q \in (x-\varepsilon, x+\varepsilon)$, i.e. contained in the open interval of width 2ε centred at x.*

Proof: By Corollary 1.2.4, we can choose $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < \varepsilon$. We now define the subset

$$X = \left\{ y \in \mathbb{N} : \frac{y}{N} > x \right\} \subseteq \mathbb{N}.$$

We know that this set is non-empty by Corollary 1.2.5. Furthermore, the set of naturals $\mathbb{N}$ is well-ordered, meaning every non-empty subset has a smallest element. Therefore, there exists a smallest element $M \in X$. This is precisely to say that

$$\frac{M-1}{N} \leq x < \frac{M}{N}, \qquad \text{which becomes} \qquad -\frac{1}{N} \leq x - \frac{M}{N} < 0$$

by adding $-\frac{M}{N}$ to all sides. If we swap the signs (i.e. multiply by -1), we can now conclude that

$$0 < \frac{M}{N} - x \leq \frac{1}{N} < \varepsilon.$$

This shows that the rational number $q := \frac{M}{N}$ lies in the subinterval $(x, x+\varepsilon) \subseteq (x-\varepsilon, x+\varepsilon)$. □

1.3 Important Inequalities

Definition Let $x \in \mathbb{R}$. Then, its absolute value is the real number $|x| \in \mathbb{R}$ given by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

Note: We can also define this as $|x| = \sup\{x, -x\}$, from which we see $x \leq |x|$ and $-x \leq |x|$.

Theorem 1.3.2 *For all $x, y \in \mathbb{R}$, the absolute value has the following properties:*

(i) $|x| \geq 0$ *with equality if and only if* $x = 0$. (**Non-Negativity**)
(ii) $|-x| = |x|$. (**Evenness**)
(iii) $|xy| = |x||y|$. (**Multiplicativity**)
(iv) $|x+y| \leq |x| + |y|$. (**Triangle Inequality**)

Proof: Both (i) and (ii) are immediate from the definition of the absolute value. Multiplicativity can be checked by considering four different cases:

- $x > 0$ and $y > 0$.
- $x > 0$ and $y < 0$.
- $x < 0$ and $y > 0$.
- $x < 0$ and $y < 0$.

Of course, if any of x and y are zero, the result is trivial. Finally, we know from the above note that $x \leq |x|$ and $y \leq |y|$; adding these inequalities tells us that $x + y \leq |x| + |y|$. Similarly, $-x \leq |x|$ and $-y \leq |y|$; adding these inequalities gives us $-(x + y) \leq |x| + |y|$. Combining these statements is precisely that $|x + y| \leq |x| + |y|$. □

Note: We can replace y with $-y$ to get an equivalent inequality to the Triangle Inequality:

$$|x - y| \leq |x| + |y|.$$

This implies the following inequality, for all $x, y, z \in \mathbb{R}$:

$$|x - y| \leq |x - z| + |y - z|.$$

1.4 Supplementary Material

Proposition (Bernoulli Inequality) *Let $x \geq 0$ and $n \in \mathbb{N}$. Then, $(1 + x)^n \geq 1 + nx$.*

Lemma (Reversed Bernoulli Inequality for $n = 1/2$) *Let $x \geq 0$. Then, $\sqrt{1 + x} \leq 1 + \frac{1}{2}x$.*

Proof: Assume to the contrary that $\sqrt{1 + x} > 1 + \frac{1}{2}x$. If we square both sides, we see that $1 + x > 1 + x + \frac{1}{4}x^2$. This implies that $0 > x^2$, which is a contradiction. □

Proposition *Let $x > 1$. Then, $x^2 > x$.*

Theorem (AM-GM Inequality) *For all $x, y \geq 0$, we have $\frac{1}{2}(x + y) \geq \sqrt{xy}$.*

Proof: The AM-GM inequality is equivalent to $\frac{1}{4}(x+y)^2 \geq xy$ (achieved by squaring both sides). Note that $(x - y)^2 \geq 0$ is always true; this is equivalent to $x^2 - 2xy + y^2 \geq 0$. Hence, we know that $x^2 + 2xy + y^2 \geq 4xy$ (by adding $4xy$ to both sides). This factorises to give $(x + y)^2 \geq 4xy$, from which we can rearrange and get the equivalent form of the AM-GM inequality. □

2 Sequences and Convergence

2.1 Sequences

Definition A sequence of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$ where we call the output a_n (instead of $a(n)$ as is normal for functions). We denote a term in the sequence by a_n and the whole sequence by $(a_n)_{n\in\mathbb{N}}$, or just (a_n) for short.

Definition 2.1.3 Let (a_n) be a sequence of real numbers.

(i) It is constant if $a_{n+1} = a_n$ for all $n \in \mathbb{N}$.
(ii) It is (monotonically) increasing if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$.
(iii) It is (monotonically) decreasing if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.
(iv) It is strictly (monotonically) increasing if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.
(v) It is strictly (monotonically) decreasing if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.
(vi) It is monotonic if it is either increasing or decreasing.
(vii) It is bounded above if there exists $M \in \mathbb{R}$ where $a_n \leq M$ for all $n \in \mathbb{N}$.
(viii) It is bounded below if there exists $L \in \mathbb{R}$ where $a_n \geq L$ for all $n \in \mathbb{N}$.
(ix) It is bounded if it is bounded above and below.

Note: A sequence (a_n) being bounded is equivalent to saying that there exists $K \in \mathbb{R}$ such that $|a_n| \leq K$ for all $n \in \mathbb{N}$. Furthermore, this is equivalent to the set $\{a_n : n \in \mathbb{N}\}$ being bounded in the sense of Definition 1.1.6

2.2 Definition of Limits

Definition 2.2.1 A real sequence (a_n) converges to a real number $L \in \mathbb{R}$ if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $|a_n - L| < \varepsilon$. In this case, we call L the limit of (a_n) and we write either $a_n \to L$ or $\lim_{n\to\infty} a_n = L$. Here, we call (a_n) convergent.

Remark Let's take a breather; Definition 2.2.1 is the first rigorous definition of a limit we have encountered, and other definitions will be built from it. Thus, it is important you have an idea of what this definition says. Indeed, we will explain it in words and we will provide a geometric interpretation in Figure 1.

(i) Given a sequence (a_n), we can show that it 'approaches' the number L as $n \in \mathbb{N}$ gets large by showing that for **any** positive number ($\varepsilon > 0$), there exists a point in the sequence a_N (there exists $N \in \mathbb{N}$) after which (for all $n > N$) every term in the sequence lies within distance that positive number of the number L ($|a_n - L| < \varepsilon$). Because this needs to work for **any** ε, the idea is that the distance can be as large or as small as you like and we should still be able to find $N \in \mathbb{N}$ to make this work.

(ii) Geometrically, this means that if we plot n against a_n on a pair of axes, then after N, every point will live inside a rectangle with width 2ε centred on the line $a_n = L$.

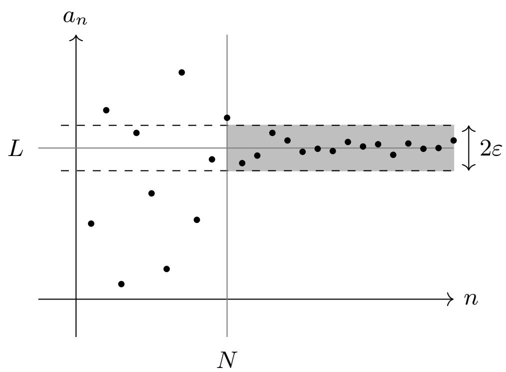

Figure 1: The geometric interpretation of the convergence of the sequence (a_n).

Note: Thus, (a_n) converges if and only if there are a **finitely-many** $a_n \notin (L - \varepsilon, L + \varepsilon)$.

Theorem 2.2.3 (Uniqueness of Limits) *The limit of a convergent sequence is unique.*

Proof: Let (a_n) be convergent and suppose that $a_n \to L$ and $a_n \to K$. We must prove $L = K$. Indeed, suppose $\varepsilon > 0$ is given. By Definition 2.2.1, we know the following:

- There exists $N_1 \in \mathbb{N}$ such that, for all $n > N_1$, $|a_n - L| < \varepsilon/2$.
- There exists $N_2 \in \mathbb{N}$ such that, for all $n > N_2$, $|a_n - K| < \varepsilon/2$.

Define $N = \max\{N_1, N_2\}$. Then, for all $n > N$, we see that

$$\begin{aligned} |L - K| &= |L - a_n + a_n - K| \\ &\leq |L - a_n| + |a_n - K|, && \text{by the Triangle Inequality,} \\ &= |a_n - L| + |a_n - K|, && \text{by properties of the absolute value,} \\ &< \varepsilon/2 + \varepsilon/2, && \text{by the inequalities above,} \\ &= \varepsilon. \end{aligned}$$

This shows that the 'distance' between the real numbers L and K is less than the positive number ε, but this works for **any** ε, so we must have $|L - K| = 0$. In other words, $L = K$ as required. □

Note: We can now talk about **the** limit of a convergent sequence. Therefore, $a_n \to L$ is equivalent to $a_n - L \to 0$, which itself is equivalent to $|a_n - L| \to 0$.

Proposition 2.2.6 (Domination) *Let (a_n) and (b_n) be sequences where $|a_n| \leq |b_n|$ for every $n \in \mathbb{N}$ and further suppose that $b_n \to 0$. Then, $a_n \to 0$.*

Proof: By assumption, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|b_n| < \varepsilon$ for all $n > N$. But then, we know that $|a_n| \leq |b_n| < \varepsilon$ for all $n > N$, which is precisely to say $a_n \to 0$. □

Proposition 2.2.7 (Shift Invariance of the Limit) *Suppose the sequence (a_n) converges to $L \in \mathbb{R}$ and let $m \in \mathbb{N}$. Then, the sequence (a_{n+m}) converges to L also.*

Proof: Let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $|a_n - L| < \varepsilon$. But $n + m > n > N$, so it also follows that $|a_{n+m} - L| < \varepsilon$, as required. □

Proposition 2.2.8 *Let (a_n) and (b_n) be sequences with $a_n \to L$ and $a_n = b_n$ for all* **except** *finitely-many $n \in \mathbb{N}$. Then, $b_n \to L$.*

Proof: Let $\varepsilon > 0$. Then, there is $N_1 \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$, when $n > N_2$. We now consider the set $\{n \in \mathbb{N} : a_n \neq b_n\}$; by assumption, this set is finite. Therefore, it has a maximal element, called N_2. Therefore, we know that $a_n = b_n$ for all $n > N_2$, where we define $N = \max\{N_1, N_2\}$. Thus, $|b_n - L| = |a_n - L| < \varepsilon$ for all $n > N$, which is to say $b_n \to L$ as required. □

2.3 Theorems About Limits

Theorem 2.3.1 *Any convergent sequence is bounded.*

Proof: Let $a_n \to L$. Then, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $|a_n - L| < 1$ (remember this works for **all** $\varepsilon > 0$, so it works for $\varepsilon = 1$ in particular). By the Triangle Inequality, this means that $|a_n| \leq |L| + 1$ for all $n > N$. Thus, define $K \coloneqq \max(\{|a_n| : n \leq N\} \cup \{|L| + 1\})$. Then, $|a_n| \leq K$ for **all** n. □

Note: It is clear that $\{|a_n| : n > N\}$ is bounded (i.e. by $|L| + 1$), but we aren't done here. It might be that an earlier term of the sequence is larger than this number; this is why we consider $\{|a_n| : n \leq N\}$. Thus, K is the largest number amongst the earlier terms of the sequence and $|L| + 1$.

Theorem 2.3.2 *Let (a_n) and (b_n) such that $a_n \to 0$ and (b_n) is bounded. Then, $a_n b_n \to 0$.*

Proof: Suppose that $K \in \mathbb{R}$ such that $|b_n| \leq K$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n| < \varepsilon/K$ whenever $n > N$. By the boundedness condition, we see that $|a_n b_n| < \varepsilon/K \cdot K = \varepsilon$ for all $n > N$; this is precisely to say $a_n b_n \to 0$. □

Theorem 2.3.3 (Algebra of Limits) *Let $a_n \to A$ and $b_n \to B$. Then, the following are true:*

(i) $a_n + b_n \to A + B$.
(ii) $a_n b_n \to AB$.
(iii) $\lambda a_n + \mu b_n \to \lambda A + \mu B$ *for any* $\lambda, \mu \in \mathbb{R}$.
(iv) $a_n / b_n \to A/B$ **if** $b_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $B \neq 0$.

Proof: (i) For each $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|a_n - A| < \varepsilon/2$ for all $n > N_1$ and $|b_n - B| < \varepsilon/2$ for all $n > N_2$. Then, for $N = \max\{N_1, N_2\}$ and $n > N$, we see that

$$\begin{aligned} |a_n + b_n - (A + B)| &= |a_n - A + b_n - B| \\ &\leq |a_n - A| + |b_n - B| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

This is precisely to say that $a_n + b_n \to A + B$.

(ii) Since (b_n) is convergent, it is bounded by Theorem 2.3.1. This means there exists $K \in \mathbb{R}$ such that $|b_n| \leq K$. Let $\varepsilon > 0$ be given and define the number

$$\varepsilon' = \frac{\varepsilon}{K + |A|} > 0.$$

Then, there exists $N_1, N_2 \in \mathbb{N}$ such that $|a_n - A| < \varepsilon'$ when $n > N_1$ and $|b_n - B| < \varepsilon'$ when $n > N_2$. Once again, let $N = \max\{N_1, N_2\}$ and suppose that $n > N$. We see that

$$\begin{aligned} |a_n b_n - AB| &= |a_n b_n - Ab_n + Ab_n - AB| \\ &= |(a_n - A)b_n - A(b_n - B)| \\ &\leq |a_n - A||b_n| + |A||b_n - B| \\ &< \varepsilon' K + |A|\varepsilon' \\ &= \varepsilon. \end{aligned}$$

This is precisely to say that $a_n b_n \to AB$.

(iii) This is an immediate consequence of (i) and (ii).

(iv) It suffices to prove that $1/b_n \to 1/B$ and then the result will follow from (ii). Indeed, let $\varepsilon > 0$ be given. Since we assume that $B \neq 0$, we know $|B|/2 > 0$. We now define the number $\varepsilon' = \varepsilon|B|^2/2 > 0$. By assumption, there exist $N_1, N_2 \in \mathbb{N}$ such that $|b_n - B| < |B|/2$ for all $n > N_1$ and $|b_n - B| < \varepsilon'$ for all $n > N_2$. Again, let $N = \max\{N_1, N_2\}$. For all $n > N$, we have

$$\begin{aligned} \left| \frac{1}{b_n} - \frac{1}{B} \right| &= \frac{|b_n - B|}{|b_n||B|} \\ &< \frac{\varepsilon'}{|B||B|/2} \\ &= \varepsilon. \end{aligned}$$

This is precisely to say that $1/b_n \to 1/B$, which suffices. □

Note: The proof of (ii) in the Algebra of Limits is different in the notes; above, we have given a direct ε-N proof but the notes uses a clever argument to avoid these gritty details.

Method – Proofs without the Algebra of Limits: Suppose we are asked to prove that a product of two sequences converges **without** using the Algebra of Limits. Then, we can just 'copy' the proof of Theorem 2.3.3 with our sequences/limits substituted into it.

Theorem 2.3.4 (Squeeze Rule) *Let $(a_n), (b_n), (c_n)$ be sequences where $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ as well as $a_n \to L$ and $c_n \to L$. Then, it follows that $b_n \to L$.*

Proof: By assumption, for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $|a_n - L| < \varepsilon$ and $|c_n - L| < \varepsilon$. By the properties of the absolute value, this means that $a_n > L - \varepsilon$ (we can rearrange $a_n - L > -\varepsilon$ to get this) and $c_n < L + \varepsilon$ (we can rearrange $c_n - L < \varepsilon$ to get this). Thus, for all $n > N$, we have

$$\begin{aligned} b_n \geq a_n &> L - \varepsilon, \\ b_n \leq c_n &< L + \varepsilon. \end{aligned}$$

Combining gives $L - \varepsilon < b_n < L + \varepsilon$, which is equivalent to $-\varepsilon < b_n - L < \varepsilon$; this is precisely $|b_n - L| < \varepsilon$. Consequently, $b_n \to L$ as required. □

Theorem 2.3.5 (Monotone Convergence Theorem) *Let (a_n) be an increasing sequence that is bounded from above. Then, (a_n) converges to $L := \sup\{a_n : n \in \mathbb{N}\}$.*

Proof: First, note that L exists by the Axiom of Completeness. Let $\varepsilon > 0$ be given. Because L is an upper bound on (a_n), we know that $a_n \leq L$ for all $n \in \mathbb{N}$. In particular, we know that $a_n \leq L < L + \varepsilon$. Because L is the **least** upper bound, we know that any number below it is **not** an upper bound. Hence, $L - \varepsilon < L$ is not an upper bound; this means there exists at least one term in the sequence larger than it (i.e. there exists $N \in \mathbb{N}$ such that $a_N > L - \varepsilon$). Because (a_n) is increasing, we know that $a_n \geq a_N$ for all $n > \mathbb{N}$; this means that $a_n > L - \varepsilon$ for all $n > N$. Combining this with the first inequality we stated gives $L - \varepsilon < a_n < L + \varepsilon$ for all $n > N$, which is precisely $|a_n - L| < \varepsilon$. □

Note: The proof of the Monotone Convergence Theorem here is of a slightly different flavour to that in the lecture notes; the same ideas are used but we do a more direct proof here. In the notes, it is framed in the context of a contradiction argument.

Corollary 2.3.6 *Let (a_n) be a decreasing sequence that is bounded from below. Then, (a_n) converges to $K := \inf\{a_n : n \in \mathbb{N}\}$.*

Proof: Define $b_n := -a_n$ and apply the Monotone Convergence Theorem (MCT) to (b_n). □

Definition Let $S \subseteq \mathbb{R}$. We say a sequence (a_n) is contained in S if $a_n \in S$ for every $n \in \mathbb{N}$.

Remark 2.3.8 Just because a sequence is contained in a set does **not** mean that the limit of that sequence (if it exists) is also contained in the set. For example, the sequence $(1/n)$ is contained in the half-open interval $S := (0, 1]$ but the limit of this sequence $0 \notin S$.

Theorem 2.3.7 *Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. Then, there exists a sequence (a_n) in S such that $a_n \to L := \sup S$.*

Proof: Because L is the least upper bound on S (by definition of the supremum), the number $L - \frac{1}{n}$ is **not** an upper bound on S. Now, using the fact that L is an upper bound implies that there exists an element of S in the interval $(L - \frac{1}{n}, L]$. Call this element a_n. Doing this for all $n \in \mathbb{N}$, we obtain a sequence (a_n). Since each term of the sequences satisfies $L - \frac{1}{n} < a_n \leq L$, we can see that $-\frac{1}{n} < a_n - L \leq 0$. The Squeeze Rule gives $a_n - L \to 0$; this means $a_n \to L$. □

Theorem 2.3.9 (Stability of Non-Negativity under Limits) *Let (a_n) be a sequence such that $a_n \geq 0$ for all $n \in \mathbb{R}$ and $a_n \to L$. Then, the limit $L \geq 0$.*

Proof: Assume to the contrary that $L < 0$. Defining $\varepsilon := -L > 0$, we see that $|x - L| < \varepsilon$ implies $x < 0$ for any $x \in \mathbb{R}$. Because $a_n \to L$, we can find some $N \in \mathbb{N}$ such that, for all $n > N$, $|a_n - L| < \varepsilon$. By our observation, this implies $a_n < 0$, a contradiction to $a_n \geq 0$ for all n. □

Corollary 2.3.10 (Stability of Closed Inequalities under Limits) *Let $a_n \to A$ and $b_n \to B$ be sequences with $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then, the limits satisfy $A \leq B$.*

Proof: Apply Theorem 2.3.9 to the sequence $(b_n - a_n)$. □

Definition 2.3.12 A subset $S \subseteq \mathbb{R}$ is called closed if **every** convergent sequence in S also has its limit in S.

Corollary 2.3.11 *Closed intervals $[a, b]$ are closed in the sense of* Definition 2.3.12.

Definition A collection of intervals $I_1, I_2, I_3 ... \subseteq \mathbb{R}$ is called nested if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

Lemma 2.3.14 (Nested Intervals Lemma) *Let (I_n) be a sequence of nested non-empty closed intervals $I_n = [a_n, b_n]$. Then, the intersection of all intervals $\cap_n I_n \neq \emptyset$.*

Proof: By the nested assumption, the sequence of lower endpoints (a_n) is increasing and the sequence of upper endpoints (b_n) is decreasing. Moreover, $a_n \leq b_m$ for all $n, m \in \mathbb{N}$. In particular, both sequences are monotonic and bounded. By the MCT, they converge. Say $a_n \to A$ and $b_n \to B$. It now follows that $a_n \leq A \leq B \leq b_n$, where we use the stability of closed inequalities to conclude that $A \leq B$. Thus, the non-empty interval $[A, B]$ is contained in every I_n. □

Definition An infinite subset $S \subseteq \mathbb{R}$ is countably infinite if there exists a bijection of the form $\mathbb{N} \to S$ (or, equivalently, of the form $S \to \mathbb{N}$).

Theorem 2.3.15 *The interval* $(0,1)$ *is* **not** *countable.*

Proof: Assume to the contrary that $(0,1)$ is countable. This means that we can find a sequence (x_n) such that for each $x \in (0,1)$, there exists a unique $n \in \mathbb{N}$ such that $x = x_n$. We now inductively construct a nested sequence of closed intervals:

I_1 is **any** closed interval in $[0,1]$ **not** containing x_1,
I_2 is **any** closed interval in I_1 **not** containing x_2,
I_3 is **any** closed interval in I_2 **not** containing x_3,

$$\vdots$$

By the Nested Intervals Lemma, there must be at least one number $y \in (0,1)$ which lives in every one of these closed intervals I_n. But by definition, $x_n \notin I_n$, so $y \neq x_n$ for any n. Hence, there does **not** exist a unique $n \in \mathbb{N}$ where $y = x_n$, a contradiction. $\square$

Corollary 2.3.16 *The set of real numbers* $\mathbb{R}$ *is not countable.*

3 Subsequences

3.1 Definition and Convergence Properties

Definition 3.1.1 A sequence (b_k) is called a subsequence of (a_n) if there exists a strictly increasing sequence of positive integers (n_k) such that $b_k = a_{n_k}$ for all $k \in \mathbb{N}$.

Note: In other words, the terms in (b_k) must occur in (a_n) in the same order. An alternate take is this: we can get (b_k) from (a_n) by deleting (possibly infinitely-many) terms.

Theorem 3.1.3 *If $a_n \to L$ and (b_k) is a subsequence of (a_n), then $b_k \to L$.*

Proof: Let $\varepsilon > 0$ be given. Since $a_n \to L$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $|a_n - L| < \varepsilon$. By the definition of a subsequence, we know that $b_k = a_{n_k}$, where (n_k) is a strictly increasing sequence of positive integers. Note that $n_1 \geq 1$ and, if $n_k \geq k$, then $n_{k+1} \geq n_k + 1 \geq k + 1$. By induction, we conclude that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore, for all $k > N$, we have $n_k \geq n_N > N$ which means $|b_k - L| = |a_{n_k} - L| < \varepsilon$. □

3.2 The Bolzano-Weierstrass Theorem

Definition Let (a_n) be a sequence. We call a term a_m dominant if every subsequent term is **not** larger than it, that is to say $a_n \leq a_m$ for all $n > m$.

Lemma 3.2.1 *Every sequence has a monotonic subsequence.*

Proof: Let (a_n) be a sequence and D be the set of dominant terms.

(i) If D is infinite, then the subsequence of dominant terms is decreasing, by definition of dominant. We have found a monotonic subsequence.

(ii) If D is finite (or empty), then there exists a term a_m beyond which there are **no** dominant terms. Let $n_1 = m+1$; since a_{n_1} is **not** dominant, there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$, but since a_{n_2} is **not** dominant, there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$, and so forth. The subsequence (a_{n_k}) is clearly increasing. We have found a monotonic subsequence. □

Theorem 3.2.2 (Bolzano-Weierstrass Theorem) *Every bounded sequence has a convergent subsequence.*

Proof: Let (a_n) be a bounded sequence. Then, there exists a monotonic subsequence (a_{n_k}) by Lemma 3.2.1. But because (a_n) is bounded, it follows that (a_{n_k}) is also bounded (by the same upper and lower bounds as the original sequence). Hence, (a_{n_k}) converges by the MCT. □

3.3 The Cauchy Property

Definition 3.3.1 A sequence (a_n) is Cauchy (or has the Cauchy property) if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n, m > N$, we have $|a_n - a_m| < \varepsilon$.

Remark The definition of Cauchy is very similar to that of convergent, with a key difference; no mention of a real number L. Instead, we look at the difference between two terms a_n and a_m. In words, where convergence is about having all terms after a certain point being within distance ε of the limit L, the Cauchy property is about having all terms after a certain point being within distance ε of **each other**.

Lemma 3.3.2 *If (a_n) is convergent, then it is Cauchy.*

Proof: Suppose $a_n \to L$ and let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $|a_n - L| < \varepsilon/2$. But then, for all $n, m > N$, we have

$$\begin{aligned} |a_n - a_m| &= |a_n - L + L - a_m| \\ &\leq |a_n - L| + |a_m - L| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

□

Lemma 3.3.3 *If (a_n) is Cauchy, then it is bounded.*

Proof: The proof is similar to that of Theorem 2.3.1. By the Cauchy property, there exists $N \in \mathbb{N}$ such that, for all $n, m > N$, we have $|a_n - a_m| < 1$. By the Triangle Inequality, this means that $|a_n| \leq |a_{N+1}| + 1$ (since the first integer m strictly greater than N is $N + 1$). We now only need to consider the maximum of the terms $|a_n|$ for $n \leq N$. Well, if we define $M := \max(\{|a_n| : n \leq N\} \cup \{|a_{N+1}| + 1\})$, we immediately see that $|a_n| \leq M$ for **all** n, which is precisely that (a_n) is bounded. □

Lemma 3.3.4 *If (a_n) is Cauchy and it has a subsequence $a_{n_k} \to L$, then $a_n \to L$.*

Proof: Let $\varepsilon > 0$ be given and consider both the convergence and the Cauchy property: there exists $N_1 \in \mathbb{N}$ such that, for all $n > N_1$, $|a_{n_k} - L| < \varepsilon/2$ and there exists $N_2 \in \mathbb{N}$ such that, for all $n, m > N_2$, $|a_n - a_m| < \varepsilon/2$. For $N = \max\{N_1, N_2\}$ and $n > N$, we have

$$\begin{aligned} |a_n - L| &= \left|a_n - a_{n_{N+1}} + a_{n_{N+1}} - L\right| \\ &\leq \left|a_n - a_{n_{N+1}}\right| + \left|a_{n_{N+1}} - L\right| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

□

Theorem 3.3.5 *A sequence (a_n) converges if and only if it is Cauchy.*

Proof: $(\Rightarrow)$ This is Lemma 3.3.2.

$(\Leftarrow)$ If (a_n) is Cauchy, then it is bounded (Lemma 3.3.3). Hence, it has a convergent subsequence (Bolzano-Weierstrass Theorem), which means (a_n) itself converges (Lemma 3.3.4). $\square$

3.4 Accumulation Points, the Limit Inferior and the Limit Superior

Definition 3.4.1 We call $K \in \mathbb{R}$ an accumulation point (or limit point or subsequential limit) of a sequence (a_n) if, for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |a_k - K| < \varepsilon\}$ is infinite.

Note: For the usual limit, we want the set $\{k \in \mathbb{N} : |a_k - L| \geq \varepsilon\}$ to be **finite**. In other words, the number of terms greater than distance ε from L is finite. As for an accumulation point, we want the number of terms **less than** the distance ε from K to be **infinite**.

Theorem 3.4.2 *The number $K \in \mathbb{R}$ is an accumulation point of (a_n) if and only if there exists a subsequence (a_{n_k}) such that $a_{n_k} \to K$.*

Proof: $(\Leftarrow)$ Suppose (a_{n_k}) is a subsequence of (a_n) such that $a_{n_k} \to K$. We must show that K is an accumulation point of (a_n). Indeed, for any $\varepsilon > 0$ given, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $|a_{n_k} - K| < \varepsilon$. Since a_{n_k} are all terms of (a_n), there are infinitely-many terms of the sequence within distance ε of K (this is precisely Definition 3.4.1).

$(\Rightarrow)$ Suppose K is an accumulation point of (a_n). We will construct a subsequence of (a_n) which converges to K. First, for $\varepsilon = 1$ the definition of an accumulation point implies there are infinitely-many terms of the sequence within distance one of K, that is infinitely-many terms where $|a_k - K| < 1$. We will chose one term of the sequence where this is the case; call it a_{n_1}. Second, repeating this with for $\varepsilon = 1/2$, we can find a term a_{n_2} which is within distance $1/2$ of K. Doing the same for $\varepsilon = 1/3$, we can find a term a_{n_3} within distance $1/3$ of K, etc. Repeating this ad infinitum, we get a subsequence (a_{n_k}) whereby $|a_{n_k} - K| < 1/k$. By the Squeeze Rule, $|a_{n_k} - K| \to 0$, which is equivalent to $a_{n_k} \to K$. $\square$

Definition Let (a_n) be bounded. Its tail is the sequence (c_k) where $c_k := \sup\{a_n : n > k\}$.

Remark By definition, (c_k) is decreasing because we are taking the supremum over a set which becomes smaller as k increases. It is also bounded, by the boundedness of (a_n), so we can apply the MCT to conclude that it also converges. This makes the next definition sensible.

Definition 3.4.3 Let (a_n) be bounded. The limit superior is $\limsup a_n := \lim_{k\to\infty} c_k$.

Note: If (a_n) is unbounded above, we define $\limsup a_n = \infty$. Note that if the sequence is bounded above **only**, then the tail is still defined and decreasing but it may be unbounded below; in this case, we define $\limsup a_n = -\infty$.

Definition 3.4.4 Let (a_n) be bounded. The limit inferior is $\liminf a_n \coloneqq -(\limsup(-a_n))$.

Remark An alternate definition of the limit inferior mirrors that of the limit superior. Indeed, we can consider the sequence $d_k \coloneqq \inf\{a_n : n > k\}$ and from it define $\liminf a_n \coloneqq \lim_{k\to\infty} d_k$.

Theorem 3.4.5 *Let (a_n) be bounded. Then, we can say the following:*
(i) $\limsup a_n$ *is the largest accumulation point of* (a_n)*.*
(ii) $\liminf a_n$ *is the smallest accumulation point of* (a_n)*.*

Proof: (i) Let $L \coloneqq \limsup a_n = \lim c_k$. Since (a_n) is bounded, we know that, for all $n \in \mathbb{N}$, $|a_n| \leq K$ for some $K > 0$. This is equivalent to $a_n \in [-K, K]$ for all n, from which we refer to Corollary 2.3.10 to conclude that $L \in [-K, K]$. As remarked above, (c_k) is bounded.

Now, assume to the contrary that $L = \limsup a_n$ is **not** an accumulation point of (a_n). Then, there exists $\varepsilon > 0$ such that the set $\{k \in \mathbb{N} : |a_k - L| < \varepsilon\}$ is finite. In other words, there are finitely-many terms $a_k \in (L - \varepsilon, L + \varepsilon)$. Given that there are only a finite number of terms in this interval, there must exist $N \in \mathbb{N}$ such that, for all $n > N$, $a_n \notin (L - \varepsilon, L + \varepsilon)$. Consequently, for $n > N$, $c_k = \sup\{a_n : n > k\} \notin (L - \varepsilon, L + \varepsilon)$, meaning that $c_k \not\to L$, a contradiction.

We have shown $L = \limsup a_n$ is an accumulation point, so it remains to show that it is the **largest** accumulation point. It suffices to show that any $M > L = \limsup a_n$ is **not** an accumulation point of (a_n). First, we choose $\varepsilon > 0$ such that $L + \varepsilon < M$. By definition, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $|c_k - L| < \varepsilon/2$ (since $L = \limsup a_n = \lim c_k$ exists). Hence,

$$c_k = c_k - L + L < \frac{\varepsilon}{2} + L < M - \frac{\varepsilon}{2},$$

where the last inequality comes by subtracting $\varepsilon/2$ from the inequality $L + \varepsilon < M$ which we had before. For $n > N$ then, we have $c_k < M - \varepsilon/2$. By the definition of c_k, it implies that $M - \varepsilon/2$ is an upper bound on (a_n) for all $n > N$. Hence, $a_n < M - \varepsilon/2$ for all $n > N$, which is equivalent to $a_n \notin (M - \varepsilon/2, M + \varepsilon/2)$ if $n > N$. This is precisely to say that M is **not** an accumulation point of (a_n).

(ii) A near-identical argument to (i) with $L \coloneqq \liminf a_n = \lim d_k$, for $d_k = \inf\{a_n : n > k\}$. □

4 Series

4.1 Definition and Convergence

Definition 4.1.1 For a series $\sum_{n=1}^{\infty} a_n$, we say that a_n is the n^{th} term of the series and we call $s_k := \sum_{n=1}^{k} a_n$ the k^{th} partial sum. We say that the series converges if the sequence of partial sums (s_k) converges in the usual sense. Otherwise, we say that the series diverges.

Proposition (Harmonic Series) *The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Proof: We will show that the sequence of partial sums (s_k) has an unbounded (and so divergent) subsequence; this will imply that (s_k) itself is divergent by the contrapositive of Theorem 3.1.3. Indeed, we consider the sequence (s_{2^p}), that is the partial sums up to $k = 2^p$:

$$\begin{aligned}
s_{2^p} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^{p-1}+1} + \cdots + \frac{1}{2^p} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{p-1}+1} + \cdots + \frac{1}{2^p}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^p} + \cdots + \frac{1}{2^p}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\
&= 1 + \frac{p}{2}.
\end{aligned}$$

This is clearly unbounded, so (s_{2^p}) diverges which implies that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. □

Proposition (Geometric Series) *For $q \in (-1, 1)$, the series $\sum_{n=1}^{\infty} q^n$ converges to $\frac{1}{1-q}$.*

Proof: We again look to the sequence of partial sums:

$$\begin{aligned}
& & s_k &= 1 + q + q^2 + \cdots + q^k \\
&\Rightarrow & q s_k &= q + q^2 + q^3 + \cdots + q^{k+1} \\
&\Rightarrow & (1-q) s_k &= 1 - q^{k+1} \\
&\Rightarrow & s_k &= \frac{1 - q^{k+1}}{1 - q}.
\end{aligned}$$

Applying the Algebra of Limits, noting that $q^{k+1} \to 0$, we conclude $s_k \to \frac{1}{1-q}$. □

Remark It is usually not possible to derive such a nice formula for s_k as was done for the geometric series proof. In fact, it would be better to develop some tests based on the sequence (a_n) of terms in the series rather than the sequence (s_k) of partial sums; this would (will) make life a bit easier.

4.2 Convergence Tests for Series

Theorem 4.2.1 (Divergence Test) *If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.*

Proof: By assumption, the sequence $s_k = \sum_{n=1}^{k} a_n$ converges to some limit $L \in \mathbb{R}$. Hence, $s_{k+1} \to L$ by Theorem 3.1.3 (since (s_{k+1}) is a subsequence). But $a_{k+1} = s_{k+1} - s_k \to L - L = 0$ by the Algebra of Limits. Since this works for all k, it must be that $a_n \to 0$. $\square$

Note: We call Theorem 4.2.1 the Divergence Test because, in its present form, it isn't the most useful (the result assumes that the series converges) but the **contrapositive** can immediately tell us if a series diverges: if $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. Be warned: it may be that $a_n \to 0$ but we still have that $\sum_{n=1}^{\infty} a_n$ diverges (e.g. the harmonic series)!

Remark If the terms of a series are all non-negative ($a_n \geq 0$ for all n), then the sequence (s_k) of partial sums is increasing. Indeed, $s_{k+1} = s_k + a_{k+1} \geq s_k$. Therefore, the MCT implies that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence (s_k) is bounded.

Notation To indicate the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded above, we will write

$$\sum_{n=1}^{\infty} a_n < \infty.$$

More specifically, if the sequence of partial sums is bounded above by $K \in \mathbb{R}$, we will write

$$\sum_{n=1}^{\infty} a_n \leq K.$$

Note: If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n \leq K$, we immediately conclude that $\sum_{n=1}^{\infty} a_n \leq K$.

Theorem 4.2.3 (Comparison Test) *Let $a_n > 0$ and $b_n > 0$ for all $n \in \mathbb{N}$.*

(i) *If a_n/b_n is bounded above and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

(ii) *If b_n/a_n is bounded above and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof: (i) By assumption, there exists $K > 0$ such that $0 \leq a_n/b_n \leq K$. This is equivalent to $0 \leq a_n \leq Kb_n$. Since $\sum_{n=1}^{\infty} b_n$ converges, this means that $\sum_{n=1}^{\infty} b_n < \infty$. Consequently, we have

$$\sum_{n=1}^{\infty} a_n \leq K \sum_{n=1}^{\infty} b_n < \infty,$$

that is $\sum_{n=1}^{\infty} a_n$ converges also.

(ii) We see that $\sum_{n=1}^{\infty} a_n$ convergent implies $\sum_{n=1}^{\infty} b_n$ convergent by (i) above. The contrapositive of this is precisely the desired statement. $\square$

Note: We call Theorem 4.2.3 the Comparison Test because the idea is to look at a series $\sum_{n=1}^{\infty} a_n$ (that we **don't** know if it converges or diverges) and ask ourselves *what familiar series does this appear like which we* **do** *know converges/diverges?*. For example, the series

$$\sum_{n=1}^{\infty} \frac{n}{2n^2 + \sin n}$$

looks complicated. However, for large n, the $\sin n$ part is irrelevant (sine takes values between 1 and -1), so the series looks roughly like $\sum_{n=1}^{\infty} n/2n^2 = \sum_{n=1}^{\infty} 1/2n$, and this is very similar to the harmonic series. Hence, we use the Comparison Test with $b_n = 1/n$.

Theorem 4.2.7 (Ratio Test) *Let $a_n > 0$ for all $n \in \mathbb{N}$ where also $a_{n+1}/a_n \to L$.*

(i) *If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.*

(ii) *If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Observation 1: if (a_n) is a sequence where $a_n \to L < 1$, then there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $a_n < 1 - \gamma$, where $\gamma := \frac{1-L}{2} > 0$. Indeed, we can take $\varepsilon = \gamma$ in the definition of convergence and note that $|a_n - L| < \gamma$ implies this fact.

Observation 2: if (a_n) is a sequence where $a_n \to L > 1$, then there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $a_n > 1 + \delta$, where $\delta := \frac{L-1}{2} > 0$. Indeed, we can take $\varepsilon = \delta$ in the definition of convergence and note that $|a_n - L| < \delta$ implies this fact.

Proof: (i) By Observation 1, there exist $N \in \mathbb{N}$ and $\gamma > 0$ such that, for all $n > N$, we have

$$0 \leq \frac{a_{n+1}}{a_n} < 1 - \gamma.$$

This is equivalent to $0 \leq a_{n+1} < (1-\gamma)a_n$. By induction, we conclude that

$$a_{N+1+n} < (1-\gamma)^n a_{N+1}$$

for all $n \in \mathbb{N}$. Therefore, we can see that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N+1} a_n + \sum_{n=N+2}^{\infty} a_n \leq \sum_{n=1}^{N+1} a_n + a_{N+1} \sum_{n=1}^{\infty} (1-\gamma)^n < \infty,$$

where we know the final sum converges by applying the geometric series formula.

(ii) By Observation 2, there exist $N \in \mathbb{N}$ and $\delta > 0$ such that, for all $n > N$, we have

$$\frac{a_{n+1}}{a_n} > 1 + \delta.$$

This is equivalent to $a_{n+1} > (1+\delta)a_n$. By induction, we conclude that

$$a_{N+1+n} > (1+\delta)^n a_{N+1}.$$

We know that $(1+\delta)^n \geq 1 + n\delta$; this is the Bernoulli Inequality. The right-hand side is unbounded above, meaning that the sequence a_{N+1+n} is unbounded. Since this is a subsequence of (a_n), it follows that (a_n) is unbounded. In particular, it doesn't converge (to zero, in particular). The Divergence Test implies that $\sum_{n=1}^{\infty} a_n$ diverges. □

Corollary (Exponentials Beat Polynomials) *Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Then, $a_n = n^k \alpha^n \to 0$.*

Sketch of Proof: The series $\sum_{n=1}^{\infty} a_n$ converges (Ratio Test), so $a_n \to 0$ (Divergence Test). □

4.3 Alternating Series

Definition 4.3.1 A series $\sum_{n=1}^{\infty} a_n$ is alternating if $a_n \neq 0$ and $a_{n+1}/a_n < 0$ for all $n \in \mathbb{N}$.

Note: In words, $a_{n+1}/a_n < 0$ means the terms swap signs $(+-+-\cdots$ or $-+-+\cdots)$.

Lemma 4.3.4 *Let (a_n) be a sequence and $L \in \mathbb{R}$. If $a_{2k} \to L$ and $a_{2k+1} \to L$, then $a_n \to L$.*

Proof: Let $\varepsilon > 0$ be given. Since $a_{2k} \to L$, there exists $K_1 \in \mathbb{N}$ such that, for all $k > K_1$, $|a_{2k} - L| < \varepsilon$. Since $a_{2k+1} \to L$, there exists $K_2 \in \mathbb{N}$ such that, for all $k > K_2$, $|a_{2k+1} - L| < \varepsilon$. Let $N := \max\{2K_1, 2K_2+1\}$. Then, for all $n > N$, either n is even (for which $n > 2K_1$ and so $|a_n - L| < \varepsilon$) or n is odd (for which $n > 2K_2 + 1$ and so $|a_n - L| < \varepsilon$). Either way, $a_n \to L$. □

Theorem 4.3.3 (Alternating Series Test) *Let (a_n) be a decreasing sequence of positive numbers which converges to zero. Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.*

Proof: Let (s_k) be the sequence of partial sums and consider the subsequence (s_{2m}):

$$\begin{aligned} s_{2m} &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2m-1} - a_{2m} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}) \\ \Rightarrow \qquad s_{2m+2} - s_{2m} &= a_{2m+1} - a_{2m+2} \\ &\geq 0. \end{aligned}$$

This implies that the sequence (s_{2m}) is increasing. Furthermore, we see that

$$\begin{aligned} s_{2m} &= a_1 - (a_2 + a_3) - \cdots - (a_{2m-2} + a_{2m-1}) - a_{2m} \\ &< a_1. \end{aligned}$$

This implies that (s_{2m}) is bounded above by a_1. By the MCT, we know that $s_{2k} \to L$, for some $L \in \mathbb{R}$. Consider now the subsequence (s_{2m+1}). But notice that $s_{2m+1} = s_{2m} + a_{2m+1}$. Applying the Algebra of Limits tells us that $s_{2m+1} \to L + 0 = L$, since $a_n \to 0$. Thus, Lemma 4.3.4 implies that $s_k \to L$ also. □

Corollary (Alternating Harmonic Series) *The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.*

Proof: This is an immediate consequence of the Alternating Series Test with $a_n = 1/n$. □

4.4 Absolute Convergence

Definition 4.4.1 The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges in the usual sense.

Theorem 4.4.2 *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.*

Proof: Let (s_k) be the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$. Then, (s_k) converges if and only if it is Cauchy (Theorem 3.3.5). Hence, it suffices to show that (s_k) is a Cauchy sequence. Now, let (t_k) be the sequence of partial sums for the series $\sum_{n=1}^{\infty} |a_n|$; we assume that this converges, which means (t_k) is Cauchy. Hence, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m > n > N$, we have $|t_m - t_n| < \varepsilon$. But notice that this says

$$|t_m - t_n| = \sum_{k=n+1}^{m} |a_k| < \varepsilon.$$

Therefore, for all $m > n > N$, we can apply the Triangle Inequality and use the above to see

$$|s_m - s_n| = \left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k| < \varepsilon,$$

by the Triangle Inequality. Hence, (s_k) is Cauchy and therefore convergent. □

Note: The converse of Theorem 4.4.2 fails; convergence doesn't imply absolute convergence. Indeed, the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ converges, but the series $\sum_{n=1}^{\infty} \left|(-1)^{n+1}/n\right| = \sum_{n=1}^{\infty} 1/n$ diverges; notice that this is the usual harmonic series.

4.5 The Importance of Absolute Convergence

Informally, absolute convergence allows one to rearrange the terms of a real series, whereas regular convergence does **not** allow for this. More precisely, if $f : \mathbb{N} \to \mathbb{N}$ is a bijection, then the series $\sum_{n=1}^{\infty} a_{f(n)}$ can be understood as a rearrangement of the series $\sum_{n=1}^{\infty} a_n$.

Theorem 4.5.2 *Let $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$ be an increasing sequence of* **finite** *subsets $S_k \subseteq \mathbb{N}$ such that their union is all the natural numbers, i.e. $\cup_{k=1}^{\infty} S_k = \mathbb{N}$. Furthermore, let (a_n) be a sequence such that $\sum_{n=1}^{\infty} a_n$ converges* **absolutely**. *Then,*

$$\lim_{k\to\infty} \sum_{n \in S_k} a_n = \sum_{n=1}^{\infty} a_n.$$

Proof: Since we know $\sum_{n=1}^{\infty} a_n$ converges absolutely, Theorem 4.4.2 implies the existence of the limit $L := \sum_{n=1}^{\infty} a_n \in \mathbb{R}$. For any $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that

$$k > N \quad \Rightarrow \quad \left| \sum_{n=1}^{k} a_n - L \right| < \frac{\varepsilon}{2},$$

by the assumption that the sequence of partial sums converges, and

$$m > k > N \quad \Rightarrow \quad \sum_{n=k+1}^{m} |a_n| < \frac{\varepsilon}{2},$$

by the assumption that the sequence of partial sums of $\sum_{n=1}^{\infty} |a_n|$ is (convergent and therefore) Cauchy. If we choose $K \in \mathbb{N}$ such that $\{1, ..., N\} \subseteq S_K$, then we get the following for all $n > K$:

$$\begin{aligned} \left| \sum_{n \in S_k} a_n - L \right| &= \left| \sum_{n=1}^{N} a_n - L + \sum_{\substack{n \in S_k \\ n > N}} a_n \right| \\ &\leq \left| \sum_{n=1}^{N} a_n - L \right| + \sum_{\substack{n \in S_k \\ n > N}} |a_n| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

□

Remark 4.5.3 In the case that each of the subsets $S_k = \{1, 2, ..., k\}$, then the left-hand side and the right-hand side of the punchline of Theorem 4.5.2 are the same, by definition.

Theorem 4.5.4 *Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent with limit $L \in \mathbb{R}$ and $f : \mathbb{N} \to \mathbb{N}$ be a bijection. Then, the series $\sum_{n=1}^{\infty} a_{f(n)}$ is absolutely convergent with the same limit L.*

Proof: This follows from Theorem 4.5.2 where $S_k = f(\{1, ..., k\})$. Then, absolute convergence follows by replacing a_n by $|a_n|$ in the statement of that result. □

Note: We can generalise things a bit. Indeed, let S be an infinite countable indexing set and $(a_\eta)_{\eta \in S}$ a family of real numbers $a_\eta \in \mathbb{R}$. Suppose also that $\sum_{n=1}^{\infty} \left|a_{f(n)}\right|$ converges. If $g : \mathbb{N} \to I$ is another bijection, we know that $f^{-1} \circ g : \mathbb{N} \to \mathbb{N}$ is bijective. Then,

$$\sum_{n=1}^{\infty} \left|a_{f(n)}\right| = \sum_{n=1}^{\infty} \left|a_{f(f^{-1}(g(n)))}\right| = \sum_{n=1}^{\infty} \left|a_{g(n)}\right|$$

and

$$\sum_{n=1}^{\infty} a_{f(n)} = \sum_{n=1}^{\infty} a_{f(f^{-1}(g(n)))} = \sum_{n=1}^{\infty} a_{g(n)}.$$

Definition 4.5.5 We say $\sum_{\eta \in S} a_\eta$ converges absolutely if $\sum_{n=1}^{\infty} a_{f(n)}$ converges absolutely in the usual sense for some (and therefore **any**) bijection $f : \mathbb{N} \to I$.

Remark An analogue of Theorem 4.5.2 holds for series as in Definition 4.5.5. The difference here is that the subsets S_k are not necessarily finite.

Theorem (Fubini's Theorem for Sums) *Let $(a_{nm})_{n,m\in\mathbb{N}}$ be a family of real numbers and assume $\sum_{n,m\in\mathbb{N}} |a_{nm}|$ converges. Then, both the sum $\sum_{m=1}^{\infty} a_{nm}$ converges absolutely for all $n \in \mathbb{N}$ and the sum $\sum_{n=1}^{\infty} a_{nm}$ converges absolutely for all $m \in \mathbb{N}$. Moreover, we have*

$$\sum_{n,m\in\mathbb{N}} a_{nm} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{nm}\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{nm}\right).$$

5 Functions and Continuity

5.1 Sequential Continuity

Definition 5.1.1 Let $D \subseteq \mathbb{R}$. Then, $f : D \to R$ is continuous at $a \in D$ if, for **all** sequences (x_n) in D where $x_n \to a$, we have $f(x_n) \to f(a)$. If f is **not** continuous at $a \in D$, we say it is discontinuous at $a \in D$. We say f is continuous if it is continuous at **every** $a \in D$.

Note: Definition 5.1.1 is sequential continuity, since it's based on convergence of sequences.

Lemma *The function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = c$ is continuous, for a fixed $c \in \mathbb{R}$.*

Proof: Let $a \in \mathbb{R}$ and $x_n \to a$. Then, $f(x_n) = c \to c = f(a)$. □

Lemma *The function $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x$ is continuous.*

Proof: Let $a \in \mathbb{R}$ and $x_n \to a$. Then, $g(x_n) = x_n \to a = g(a)$. □

Lemma *The function $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ where $h(x) = \frac{1}{x}$ is continuous.*

Proof: Let $a \in \mathbb{R} \setminus \{0\}$ and $x_n \to a$. Then, $h(x_n) = \frac{1}{x_n} \to \frac{1}{a} = h(a)$ by the Algebra of Limits. □

Proposition *Every function $f : \mathbb{Z} \to \mathbb{R}$ is continuous.*

Proof: Let $a \in \mathbb{Z}$ and (x_n) be a sequence in $\mathbb{Z}$ where $x_n \to a$. By definition of convergence, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $|x_n - a| < 1/2$. However, since $x_n \in \mathbb{Z}$ for every n, this tells us that the difference between two integers is less than $1/2$; this means the integers are equal. Thus, for any $\varepsilon > 0$ and $n > N$, we have $|f(x_n) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon$. Consequently, $f(x_n) \to f(a)$. □

Theorem 5.1.3 (ε-δ Continuity) *A function $f : D \to \mathbb{R}$ is continuous at $a \in D$ if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in D$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \varepsilon$.*

Proof: ($\Leftarrow$) Suppose that f satisfies the ε-δ definition of continuity. We must show that f is continuous in the usual sense of Definition 5.1.1. Well, let (x_n) be a sequence in D where $x_n \to a$. By the ε-δ property, there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ **if** $|x - a| < \delta$. Since $x_n \to a$, we can choose $N \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n > N$. Therefore, we will get for free that $|f(x_n) - f(a)| < \varepsilon$ when $n > N$. This is precisely to say $f(x_n) \to f(a)$, as required.

(⇒) We prove the contrapositive, so suppose that f **doesn't** satisfy the ε-δ property. Then, there exists $\varepsilon > 0$ such that, for all $\delta > 0$, there is some $x \in D$ with $|x - a| < \delta$ such that $|f(x) - f(a)| \geq \varepsilon$. In particular, let $\delta = 1/n$ and let $x_n \in D$ be the element corresponding to x where $|x_n - a| < 1/n$. Clearly, the sequence $x_n \to a$ but $f(x_n) \not\to f(a)$ since $|f(x_n) - f(a)| \geq \varepsilon$. Hence, f is **not** continuous. □

Remark We explain Theorem 5.1.3 in words and provide a geometric interpretation in Figure 2.

(i) Given a function $f : D \to \mathbb{R}$, we can show that it is continuous at $a \in D$ by showing that for **any** positive number ($\varepsilon > 0$), there exists a small distance ($\delta > 0$) such that if we move the **input** of the function away from a by less than that distance (for all $x \in D$ with $|x - a| < \delta$), then the **output** of the function will move away from $f(a)$ by less than the given positive number ($|f(x) - f(a)| < \varepsilon$).

(ii) Geometrically, the graph of $f(x)$ should stay inside the rectangle for all x in the rectangle.

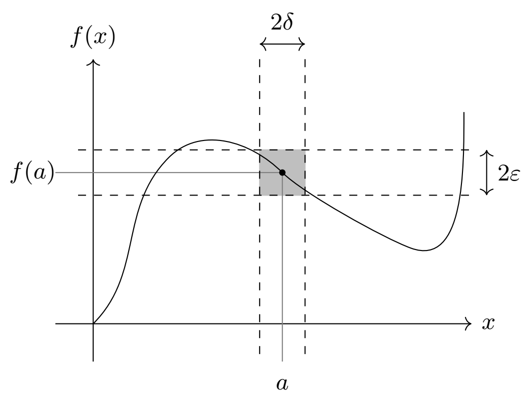

Figure 2: The geometric interpretation of continuity of $f : D \to \mathbb{R}$.

5.2 Basic Properties of Continuous Functions

Theorem 5.2.1 (Algebra Property of Continuous Functions) *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous at $a \in D$. Then, the following are true:*

(i) *$f + g$ is continuous at a.*

(ii) *fg is continuous at a.*

(iii) *$1/f$ is continuous at a* **if** *$f(x) \neq 0$ for all $x \in D$.*

Proof: Let (x_n) be a sequence in D such that $x_n \to a$. By assumption, we know $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$. Consequently, the Algebra of Limits implies the following:

$$\begin{aligned} (f+g)(x_n) &= f(x_n) + g(x_n) \to f(a) + g(a) = (f+g)(a), \\ (fg)(x_n) &= f(x_n)g(x_n) \quad \to f(a)g(a) \quad = (fg)(a), \\ (1/f)(x_n) &= 1/f(x_n) \quad \to 1/f(a) \quad = (1/f)(a). \end{aligned}$$

Thus, $f + g$ and fg and $1/f$ (so long as f is non-zero on D) are continuous at a. □

Definition A polynomial in x is a function of the form $p(x) = a_0 + a_1x + \cdots + a_nx^n$ where the coefficients $a_0, ..., a_n \in \mathbb{R}$. If $a_n \neq 0$, we say that the degree of p is n, denoted $\deg(p)$.

Proposition 5.2.2 *Every polynomial function $p : \mathbb{R} \to \mathbb{R}$ is continuous.*

Proof: We prove this by induction on the degree of the polynomial, namely n. Indeed, the base case is where $n = 0$, so the polynomial is constant and is therefore continuous. Next, assume that every polynomial of degree k is continuous and let p be a polynomial of degree $k+1$. Hence, we can write $p(x) = xq(x) + a_0$, where q is a polynomial of degree k. By assumption, q is continuous; we know that the polynomial $g(x) = x$ is continuous from earlier. Hence, the Algebra Property of Continuous Functions implies that p is continuous. By induction, the result holds for all polynomials. $\square$

Corollary *Let p and q be polynomials and $D = \{x \in \mathbb{R} : q(x) \neq 0\}$. Then, the function $f : D \to \mathbb{R}$ given by $f(x) = p(x)/q(x)$ is continuous. We call this a* rational function in x.

Note: Where the coefficients live in $\mathbb{R}$, we denote the set of polynomials in one variable by $\mathbb{R}[x]$. Furthermore, we denote the set of rational functions in one variable by $\mathbb{R}(x)$.

Theorem 5.2.3 *Let $D, E \subseteq \mathbb{R}$ and $f : D \to E$ be continuous at $a \in D$ and $g : E \to \mathbb{R}$ be continuous at $f(a) \in E$. Then, the function $g \circ f : D \to \mathbb{R}$ is continuous at a.*

Proof: Let (x_n) be a sequence in D with $x_n \to a$. By the continuity of f at a, $f(x_n) \to f(a)$. By the continuity of g at $f(a)$, $(g \circ f)(x_n) = g(f(x_n)) \to g(f(a)) = (g \circ f)(a)$. This is precisely to say that $g \circ f$ is continuous at a. $\square$

5.3 The Intermediate Value Theorem

Theorem 5.3.1 (Intermediate Value Theorem) *Let $f : [a, b] \to \mathbb{R}$ be continuous and $y \in \mathbb{R}$ be a number between $f(a)$ and $f(b)$. Then, there exists $c \in [a, b]$ such that $f(c) = y$.*

Proof: If $f(a) = f(b)$, the result is trivial as $y = f(a)$ is the only possibility; just choose $c = a$. We will now split into cases, although the second will follow immediately from the first.

(i) Assume that $f(a) < f(b)$. Define the set $X := \{x \in [a, b] : f(x) \leq y\}$. Certainly X is non-empty because $a \in X$ and it is bounded above by b. Therefore, the Axiom of Completeness implies that $c := \sup(X)$ exists. Because the interval $[a, b]$ is closed, we know that the supremum $c \in [a, b]$. Now, let (x_n) be a sequence in X such that $x_n \to c$ (this exists by Theorem 2.3.7). It follows that $f(x_n) \leq y$. By the continuity of f, we know that $f(x_n) \to f(c)$. As a consequence of the stability of closed inequalities under limits (Corollary 2.3.10), we know that $f(c) \leq y < f(b)$. Next, because c is the supremum of

X, we know that $c + 1/n$ is **not** in X. Because $c < b$, we know that $c + 1/n \in [a, b]$ for n sufficiently large. Therefore, for large enough n, we know that $f(c + 1/n) > y$. Because $c + 1/n \to c$, we know that $f(c + 1/n) \to f(c) \geq y$. Hence, $f(c) = y$.

(ii) Assume that $f(a) > f(b)$. Then, define $g = -f$ and apply (i) above to g. □

Definition 5.3.4 A function $f : D \to \mathbb{R}$ is called strictly increasing if, for all $x, y \in D$ with $x < y$, we have $f(x) < f(y)$.

Lemma 5.3.5 *Let $k \in \mathbb{N}$. Then, $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = x^k$ is strictly increasing.*

Proof: Let $y > x > 0$. Then, we can factorise the following:

$$f(y) - f(x) = y^k - x^k = (y - x)(y^{k-1} + y^{k-2}x + \cdots + yx^{k-2} + x^{k-1}).$$

But $y - x > 0$ and the second factor $y^{k-1} + y^{k-2}x + \cdots + yx^{k-2} + x^{k-1} \geq y^{k-1} > 0$. This implies that $f(y) - f(x) > 0$, precisely what was to be proved. □

Proposition 5.3.6 (Existence of Roots) *Let $k \in \mathbb{N}$. Given any $y \geq 0$, there exists a unique $x \geq 0$ such that $x^k = y$. We denote this real number by $x^{1/k}$ and call it the k^{th} root of y.*

Proof: For **existence**, let $f : [0, 1 + y] \to \mathbb{R}$ be given by $f(x) = x^k$. This is continuous as a result of Proposition 5.2.2. Note that $f(0) = 0 \leq y$. Additionally, notice the following:

(i) If $y \geq 1$, then $f(1 + y) > f(y) = y^k \geq y$; this uses that x^k is increasing.

(ii) If $y < 1$, then $f(1 + y) \geq f(1) = 1 > y$; this uses that x^k is increasing.

Either way, we see that $f(1 + y) > y$. This means that y is a real number between $f(0)$ and $f(1+y)$. By the Intermediate Value Theorem (IVT), there exists $x \in [0, 1+y]$ such that $f(x) = y$, that is $x^k = y$. For **uniqueness**, let $z \in [0, \infty)$ also satisfy $f(z) = y$. Then, $f(z) = f(x)$. This means that $z \not< x$ (otherwise it would contradict Lemma 5.3.5) and $z \not> x$ (since this would again contradict Lemma 5.3.5). Therefore, $z = x$. □

Lemma 5.3.7 *The function $g : [0, \infty) \to \mathbb{R}$ given by $g(x) = x^{1/k}$ is strictly increasing.*

Proof: Assume to the contrary that $0 \leq x < y$ but $g(x) \geq g(y)$ and define $f(x) \coloneqq x^k$. Then, by Lemma 5.3.5, $f(g(x)) \geq f(g(y))$. However, $f(g(x)) = x$ and $f(g(y)) = y$, which means that $x \geq y$, a contradiction. □

Proposition 5.3.8 *The function $g : [0, \infty) \to \mathbb{R}$ given by $g(x) = \sqrt{x} \coloneqq x^{1/2}$ is continuous.*

Proof: (Continuity at $a > 0$) Let $a \in (0, \infty)$ and (x_n) be a sequence in $[0, \infty)$ where $x_n \to a$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $|x_n - a| < \varepsilon\sqrt{a}$, which is a positive real number. Therefore, for all $n > N$,

$$\begin{aligned} |g(x_n) - g(a)| &= \left|\sqrt{x_n} - \sqrt{a}\right| \\ &= \frac{|x_n - a|}{\sqrt{x_n} + \sqrt{a}} \\ &< \frac{|x_n - a|}{\sqrt{a}} \\ &< \varepsilon. \end{aligned}$$

Therefore, $g(x_n) \to g(a)$, so g is continuous at every $a > 0$.

(Continuity at $a = 0$) Let (x_n) be a sequence in $[0, \infty)$ where $x_n \to 0$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we see that $0 \leq |x_n - 0| = x_n < \varepsilon^2$. Therefore, for all $n > N$,

$$\begin{aligned} |g(x_n) - 0| &= |\sqrt{x_n}| \\ &= \sqrt{x_n} \\ &< \sqrt{\varepsilon^2} \\ &= \varepsilon. \end{aligned}$$

Hence, $g(x_n) \to 0 = g(0)$, so g is continuous at zero also. □

Proposition 5.3.9 *The function $g : [0, \infty) \to \mathbb{R}$ given by $g(x) = x^{1/k}$ is continuous.*

Proof: (Continuity at $a > 0$) Suppose $a \in (0, \infty)$ and (x_n) is a sequence in $[0, \infty)$ where $x_n \to a$. Let $\varepsilon > 0$ be given. The trick is to use the factorisation formula in the proof of Lemma 5.3.5. To this end, define $y_n := g(x_n) = x_n^{1/k}$ and $b := g(a) = a^{1/k}$. Then,

$$\begin{aligned} & y_n^k - b^k = (y_n - b)(y_n^{k-1} + y_n^{k-2}b + \cdots + y_n b^{k-2} + b^{k-1}) \\ \Rightarrow \quad & y_n - b = \frac{y_n^k - b^k}{y_n^{k-1} + y_n^{k-2}b + \cdots + y_n b^{k-2} + b^{k-1}} \\ \Rightarrow \quad & |y_n - b| \leq \frac{\left|y_n^k - b^k\right|}{b^{k-1}} \\ \Leftrightarrow \quad & |g(x_n) - g(a)| \leq \frac{|x_n - a|}{b^{k-1}}. \end{aligned}$$

Since $x_n \to a$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $|x_n - a| < \varepsilon b^{k-1}$. Hence, for $n > N$, the above estimate implies that $|g(x_n) - g(a)| < \varepsilon$. We conclude therefore that $g(x_n) \to g(a)$, meaning g is continuous at every $a > 0$.

(Continuity at $a = 0$) Suppose (x_n) is a sequence in $[0, \infty)$ where $x_n \to 0$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $|x_n - 0| = x_n < \varepsilon^k$. Therefore, for all $n > N$,

$$|g(x_n) - 0| = x_n^{1/k} < \varepsilon.$$

Hence, $g(x_n) \to 0 = g(0)$, so g is continuous at zero also. □

5.4 The Extreme Value Theorem

Definition 5.4.1 Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say that f is bounded above if there exists $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in D$. In this case, we define the supremum of the function, $\sup(f)$, to be its least upper bound. Similarly, we say that f is bounded below if there exists $L \in \mathbb{R}$ such that $f(x) \geq L$ for all $x \in D$. In this case, we define the infimum of the function, $\inf(f)$, to be its greatest lower bound.
If there exists $a \in D$ such that $f(x) \leq f(a) = M$ for all $x \in D$, we say that f attains a maximum value of M at a. Similarly, if there exists $b \in D$ such that $f(x) \geq f(b) = L$ for all $x \in D$, we say that f attains a minimum value of L at b.

Note: As expected, we say the function f is bounded if it is bounded above and below.

Remark A function may be bounded but **not** attain a maximum value. It will, however, still have a supremum (guaranteed by the Axiom of Completeness). For example, the function $f(x) = -1/x$ is bounded above by 0, but it never attains this bound. Conversely, **if** a function attains a maximum value, it is automatically bounded above and its maximum value is its supremum.

Theorem 5.4.2 *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, f is bounded above. Furthermore, there exists $c \in [a, b]$ such that $f(c) = \sup(f)$.*

Proof: Assume to the contrary that f is unbounded above. Then for each $n \in \mathbb{N}$, n is **not** an upper bound on f, so there exists a number $x_n \in [a, b]$ such that $f(x_n) > n$. This defines a sequence (x_n) which is bounded; the Bolzano-Weierstrass Theorem implies the existence of a convergent subsequence (x_{n_k}), say $x_{n_k} \to x$. Because $a \leq x_{n_k} \leq b$ for all $k \in \mathbb{N}$, we know the limit $a \leq x \leq b$ (Corollary 2.3.10). Since f is continuous, we know that $f(x_{n_k}) \to f(x)$. Because convergent sequences are bounded (Theorem 2.3.1), we know that $f(x_{n_k})$ is bounded. However, we assumed that $f(x_{n_k}) > n_k \geq k$, so $f(x_{n_k})$ is unbounded above; this is a contradiction.

The image of f, which is the set $\mathrm{im}(f) := \{f(t) : t \in [a, b]\}$, is non-empty and bounded above (which is what we just proved by contradiction), it has a supremum, M say. Well, for any $n \in \mathbb{N}$, we see that $M - 1/n < M$, meaning $M - 1/n$ is **not** an upper bound on $\mathrm{im}(f)$ (and therefore f). Hence, there exists $y_n \in [a, b]$ such that $f(y_n) \geq M - 1/n$. Combining this with the definition of the supremum, we know that

$$M - \frac{1}{n} \leq f(y_n) \leq M.$$

Since (y_n) is bounded, it has a convergent subsequence by the Bolzano-Weierstrass Theorem (this is the same argument used in the contradiction done above), say (y_{n_k}) such that $y_{n_k} \to c$. By the Squeeze Rule applied to the inequalities above, we see that

$$M \leq f(c) \leq M,$$

where we have used the continuity of f to conclude $f(y_{n_k}) \to f(c)$. Thus, $f(c) = M = \sup(f)$. □

Corollary 5.4.3 (Extreme Value Theorem) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then, f is bounded and attains both a maximum value and a minimum value.*

Proof: The fact f is bounded above and attains a maximum value is simply Theorem 5.4.2. To see it is bounded below and attains a minimum value, define $g : [a, b] \to \mathbb{R}$ by $g(x) = -f(x)$ and apply Theorem 5.4.2 to the function g. □

5.5 Supplementary Material

Remark Continuity is a so-called **local property** in this sense: to check if a function $f : D \to \mathbb{R}$ is continuous at $a \in D$, it is sufficient to know only how f behaves 'near' a; we make this rigorous.

Lemma *Let $D \subseteq \mathbb{R}$, $a \in D$ and $f : D \to \mathbb{R}$ be a function. Suppose $\delta > 0$ is any positive number and the function g is obtained by restricting f onto $(a - \delta, a + \delta) \cap D$. Then, f is continuous at a if and only if g is continuous at a.*

Sketch of Proof: Any sequence (x_n) in D such that $x_n \to a$ will eventually end up being contained in the set $(a - \delta, a + \delta) \cap D$. In other words, there exists $N \in \mathbb{N}$ such that $x_n \in (a - \delta, a + \delta) \cap D$ for all $n > N$. □

Proposition *The absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ is continuous.*

Proof: The absolute value function $x \mapsto |x|$ is the same as the function $x \mapsto x$ on the set $[0, \infty)$ and the same as the function $x \mapsto -x$ on $(-\infty, 0)$. Both of these functions are continuous, so the previous lemma implies that $|\cdot|$ is continuous at every $a \in \mathbb{R} \setminus \{0\}$. Some special care is taken for continuity at $a = 0$. Indeed, consider an arbitrary sequence $x_n \to 0$. Then, $|x_n| \to 0$, so it is indeed also continuous at zero. □

Definition Let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is called piecewise if it is given by

$$f(x) = \begin{cases} g_1(x), & \text{if } x \in X_1 \\ g_2(x), & \text{if } x \in X_2 \\ \quad\vdots & \\ g_k(x), & \text{if } x \in X_k \end{cases},$$

where the $X_i \subseteq D$ are pairwise disjoint ($X_i \cap X_j = \emptyset$ for all $i \neq j$) and $D = \bigcup_{i=1}^{k} X_i$.

This definition is pretty general, but we will only really be concerned with piecewise functions $f : \mathbb{R} \to \mathbb{R}$ of the following form, where $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous:

$$f(x) = \begin{cases} g(x), & \text{if } x \geq 0 \\ h(x), & \text{if } x < 0 \end{cases}.$$

By definition, this means that f agrees with g on the interval $(0, \infty)$ and f agrees with h on the interval $(-\infty, 0)$. The only place we need check now is $x = 0$.

> **Proposition** *For the piecewise function $f : \mathbb{R} \to \mathbb{R}$ with two 'pieces' above, f is continuous at zero if $g(0) = h(0)$ and discontinuous at zero otherwise.*

Proof: (Continuous Case) Assume $g(0) = h(0)$. Consider an arbitrary sequence $x_n \to 0$; we will show that $f(x_n)$ has but one accumulation point and is therefore bounded (and thus converges). Because (x_n) is convergent, it is bounded (Theorem 2.3.1). Therefore, there exists $M \in \mathbb{N}$ such that $x_n \in [-M, M]$ for all n. By the Extreme Value Theorem (EVT), we see that both g and h are bounded when their domains are restricted to $[-M, M]$. This therefore means that f is bounded on $[-M, M]$, since $\sup(f) \leq \max\{\sup(g), \sup(h)\}$ and $\inf(f) \geq \max\{\inf(g), \inf(h)\}$. We can conclude that the sequence $f(x_n)$ is bounded; we know it has a convergent subsequence $f(x_{n_k})$ by the Bolzano-Weierstrass Theorem. There are two possibilities.

(i) There are infinitely-many terms $x_{n_k} \geq 0$. In this case, we have a subsequence that converges to zero **from the right**; this means that $f(x_{n_k}) \to g(0)$.

(ii) There are infinitely-many terms $x_{n_k} < 0$. In this case, we have a subsequence that converges to zero **from the left**; this means that $f(x_{n_k}) \to h(0)$.

As $g(0) = g(0)$, there is one accumulation point. This implies $\liminf f(x_n) = f(0) = \limsup f(x_n)$. Consequently, $f(x_n) \to f(0)$, which is precisely to say f is continuous at zero.

(Discontinuous Case) Assume $g(0) \neq h(0)$. To show that f is **not** continuous at zero, it is enough to give **one** sequence (x_n) where $x_n \to 0$ but $f(x_n) \not\to f(0)$. Indeed, consider $x_n = (-1)^n/n \to 0$. By the fact that g and h are continuous, we know that $g(x_n) \to g(0)$ and $h(x_n) \to h(0)$. We see that $x_{2k} > 0$ and $x_{2k+1} < 0$, for all $k \in \mathbb{N}$. Therefore,

$$f(x_n) = \begin{cases} g(x_n), & \text{if } n \text{ is even} \\ h(x_n), & \text{if } n \text{ is odd} \end{cases}.$$

As such, $f(x_n)$ has two convergent subsequences: $f(x_{2k}) \to g(0)$ and $f(x_{2k+1}) \to h(0)$. Because $g(0) \neq h(0)$, it follows that $f(x_n)$ is **not** convergent and thus f is discontinuous at zero. $\square$

6 Some Symbolic Logic

6.1 Statements and their Symbolic Manipulation

Definition A statement is, for our purposes, any declaration which is unambiguously true or false. We denote a statement by a capital letter, say P. Suppose we have two statements P and Q. We can consider new statements from them:

- The negation $\neg P$ is true if and only if P is false.
- The conjunction $P \wedge Q$ is true if and only if **both** P and Q are true.
- The disjunction $P \vee Q$ is true if and only if **at least one** of P and Q is true.

Note: When a mathematician says "or", they mean "this or that **or both**". Also then, we can think of the new statements in the above definition in words as follows:

(i) $\neg P$ is "not P".
(ii) $P \wedge Q$ is "P and Q".
(iii) $P \vee Q$ is "P or Q (or both)".

Definition A truth table represents the truth/falsity of a constructed statement in terms of the truth/falsity of its constituent pieces. We represent "false" by 0 and "true" by 1.

Method – Logically-Equivalent Statements: To show that two statements are logically equivalent (meaning one is true if and only if the other is true), it suffices to construct a truth table and exhibit that the columns representing each of the statements are identical. In this case, we denote logical equivalence by the symbol $\Leftrightarrow$.

Lemma *For a statement P, we have the logical equivalence $P \Leftrightarrow \neg(\neg P)$.*

Proof: We can easily show this by constructing the truth table:

P	$\neg P$	$\neg(\neg P)$
0	1	0
1	0	1

Table 1: The truth table for $P \Leftrightarrow \neg(\neg P)$.

Indeed, we see that the columns for P and for $\neg(\neg P)$ are the same. □

Theorem (de Morgan's Laws) *For statements P and Q, we have these logical equivalences:*

(i) $\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$.
(ii) $\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$.

Proof: (i) We can prove the first law by constructing another truth table

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

Table 2: The truth table for $\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$.

We see that the columns for $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are the same.

(ii) We can again construct a truth table, or we can use the first law with the previous lemma:

$$\begin{aligned} \neg(P \vee Q) &\Leftrightarrow \neg(\neg(\neg P) \vee \neg(\neg Q)) \\ &\Leftrightarrow \neg\big(\neg\big((\neg P) \wedge (\neg Q)\big)\big) \\ &\Leftrightarrow (\neg P) \wedge (\neg Q), \end{aligned}$$

where we applied the first of de Morgan's Laws to the statements $\neg P$ and $\neg Q$ to get from the first line to the second line (i.e. the second logical equivalence). □

Corollary *For statements P, Q and R, we have the logical equivalences*

$$\begin{aligned} \neg(P \vee (Q \wedge R)) &\Leftrightarrow (\neg P) \wedge \neg(Q \wedge R) \\ &\Leftrightarrow (\neg P) \wedge ((\neg Q) \vee (\neg R)). \end{aligned}$$

6.2 Implications

Definition An implication is a new statement build from two others, P and Q. It is denoted by $P \Rightarrow Q$ and spoken as "if P, then Q" or "P implies Q".

Note: The standard notation for an implication is actually $P \to Q$ but this conflicts with our notation of convergence, so we stick with $P \Rightarrow Q$ from here. Its truth table is below.

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Table 3: The truth table for $P \Rightarrow Q$.

Lemma *For statements P and Q, we have the logical equivalence $(P \Rightarrow Q) \Leftrightarrow (\neg P) \vee Q$.*

Sketch of Proof: Simply construct the truth table for $(\neg P) \vee Q$. □

Definition Suppose we have statements P and Q and an implication $P \Rightarrow Q$.

- The converse is the statement $Q \Rightarrow P$.
- The contrapositive is the statement $\neg Q \Rightarrow \neg P$.

Proposition *For statements P and Q, we have the logical equivalence*

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P).$$

Sketch of Proof: Simply construct the truth table for $\neg Q \Rightarrow \neg P$. □

Note: The negation of an implication is **not** another implication. Indeed, we see from the previous lemma $\neg(P \Rightarrow Q) \Leftrightarrow \neg((\neg P) \vee Q) \Leftrightarrow P \wedge (\neg Q)$; the latter is not an implication.

6.3 Quantifiers

Definition Let $P(x)$ be a statement making sense for all $x \in A$. Here, A is just some set.

- The statement $\forall x\, P(x)$ means "for all $x \in A$, $P(x)$ is true".
- The statement $\exists x\, P(x)$ means "there exists $x \in A$ such that $P(x)$ is true".

We call $\forall$ the universal quantifier and $\exists$ the existential quantifier.

Proposition *Let $P(x)$ be true for all $x \in A$. Then, we have the following:*

(i) $\neg(\forall x\, P(x)) \Leftrightarrow \exists x\, \neg P(x)$.
(ii) $\neg(\exists x\, P(x)) \Leftrightarrow \forall x\, \neg P(x)$.

Note: In words, (i) means that a statement is **false** for **all** values in a set if and only if there exists a single value for which the statement fails to be true. Also, (ii) means there is **no** value making the statement **true** if and only if **all** values make the statement false.

Theorem *The statement "the sequence (a_n) does* **not** *converge to $L \in \mathbb{R}$" means that there exists $\varepsilon > 0$ such that, for all $N \in \mathbb{N}$, there exists $n > N$ with $|a_n - L| \geq \varepsilon$.*

Sketch of Proof: Write out the definition of $a_n \to L$ using quantifiers and negate it. □

7 Limits of Functions

7.1 The Main Definition

Definition 7.1.1 Let $D \subseteq \mathbb{R}$. Then, a point $a \in D$ is called a limit point of D if there exists a sequence (x_n) in $D \setminus \{a\}$ such that $x_n \to a$.

Definition 7.1.2 Let $D \subseteq \mathbb{R}$, $a \in D$ a limit point of D and $f : D \to \mathbb{R}$. We say that f has a limit $L \in \mathbb{R}$ at a if, for all sequences (x_n) in $D \setminus \{a\}$ where $x_n \to a$, we have $f(x_n) \to L$. In this case, we write $\lim\limits_{x \to a} f(x) = L$ (the left-hand side is often $\lim_{x \to a} f(x)$ when typed).

Note: In Definition 7.1.2, every term $x_n \neq a$. Thus, the value $f(a)$ is irrelevant, if it exists.

Proposition 7.1.3 (Uniqueness of Limits) *If it exists, $\lim_{x \to a} f(x)$ is unique.*

Proof: Let $f : D \to \mathbb{R}$ and assume that $L, K \in \mathbb{R}$ both satisfy the definition of $\lim_{x \to a} f(x)$. For any sequence (x_n) in $D \setminus \{a\}$ such that $x_n \to a$, we know that $f(x_n) \to L$ and $f(x_n) \to K$. By the uniqueness of limits of sequences (Theorem 2.2.3), we know that $L = K$. □

Theorem 7.1.5 (Algebra of Limits) *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be functions with limits $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = K$ Then, the following are true:*

(i) $\lim_{x \to a}(f(x) + g(x)) = L + K$.
(ii) $\lim_{x \to a}(f(x)g(x)) = LK$.
(iii) $\lim_{x \to a} 1/f(x) = 1/L$ **if** $f(x) \neq 0$ *for all* $x \in D$ *and* $L \neq 0$.

Proof: Let (x_n) be a sequence in $D \setminus \{a\}$ where $x_n \to a$. Then, $f(x_n) \to L$ and $g(x_n) \to K$ by assumption. Hence, the Algebra of Limits of sequences (Theorem 2.3.3) implies the result:

(i) $f(x_n) + g(x_n) \to L + K$.

(ii) $f(x_n)g(x_n) \to LK$.

(iii) $1/f(x_n) \to 1/L$ **if** $f(x_n) \neq 0$ for all $n \in \mathbb{N}$ and $L \neq 0$. □

7.2 An Alternative Definition for Limits of Functions

Theorem 7.2.1 (ε-δ Limits) *A function $f : D \to \mathbb{R}$ has a limit $L \in \mathbb{R}$ at $a \in D$ if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in D$ with $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.*

Proof: ($\Leftarrow$) Suppose that f satisfies the ε-δ definition of having a limit L at a. We must show that f has a limit in the usual sense of Definition 7.1.2. Well, let (x_n) be a sequence in $D \setminus \{a\}$

where $x_n \to a$. By the ε-δ property, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ **if** $0 < |x - a| < \delta$. Since $x_n \to a$, we can choose $N \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n > N$. Because every $x_n \neq a$, we have also that $0 < |x_n - a| < \delta$ for all $n > N$. Thus, we have $|f(x_n) - L| < \varepsilon$ when $n > N$

($\Rightarrow$) We prove the contrapositive, so suppose that f **doesn't** satisfy the ε-δ property. Then, there exists $\varepsilon > 0$ such that, for all $\delta > 0$, there is some $x \in D$ with $0 < |x - a| < \delta$ such that $|f(x) - L| \geq \varepsilon$. In particular, let $\delta = 1/n$ and let $x_n \in D$ be the element corresponding to x such that $0 < |x_n - a| < 1/n$. It is clear that (x_n) is a sequence in $D \setminus \{a\}$ where $x_n \to a$ but $f(x_n) \not\to L$ since $|f(x_n) - L| \geq \varepsilon$. Hence, f does **not** have a limit L at a. □

Corollary 7.2.2 *For $D \subseteq \mathbb{R}$ and $a \in D$, $f : D \to \mathbb{R}$ is continuous at a if and only if* **either**

(i) *a is* **not** *a limit point of D (we call a an isolated point in this case),* **or**

(ii) *the limit of f at a exists and is equal to its value at a, that is $\lim_{x\to a} f(x) = f(a)$.*

Proof: This is immediate by comparing Theorem 7.2.1 to Theorem 5.1.3 and Definition 7.1.1. □

7.3 Limits at Infinity

Definition 7.3.1 A real sequence (a_n) diverges to infinity if, for each $K \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $a_n > K$. In this case, we write $a_n \to \infty$. Similarly, (a_n) diverges to minus infinity if, for each $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $a_n < M$. In this case, we write $a_n \to -\infty$.

Remark We explain Definition 7.3.1 in words and with a geometric interpretation in Figure 3.

(i) Given a sequence (a_n), we can show that it diverges to infinity by showing that for **any** number ($K \in \mathbb{R}$), there exists a point in the sequence a_N (there exists $N \in \mathbb{N}$) after which (for all $n > N$) every term in the sequence exceeds that number ($a_n > K$).

(ii) Geometrically, this means that if we plot n against a_n on a pair of axes, then after N, every point will live **above** the line $a_n = K$. Similarly, divergence to negative infinity means that every point for $n > N$ will live **below** the line $a_n = M$.

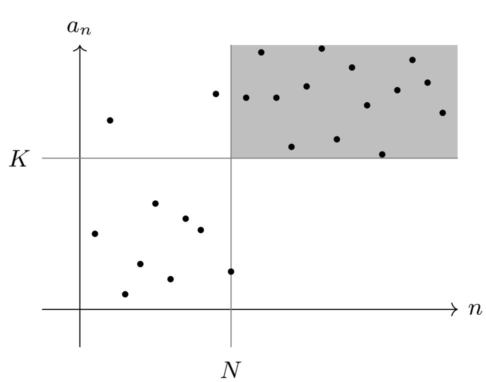

Figure 3: The geometric interpretation of the divergence of (a_n) to infinity.

Note: It is clear that the sequence $a_n \to -\infty$ if and only if $-a_n \to \infty$.

Proposition 7.3.5 (Sweep Rule) *If $a_n \to \infty$ and $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $b_n \to \infty$.*

Proof: Let $K \in \mathbb{R}$ be given. Since $a_n \to \infty$, there exists $N \in \mathbb{N}$ such that $a_n > K$ for all $n > N$. But it follows that $b_n \geq a_n > K$ for all $n > N$, so $b_n \to \infty$ also. □

Proposition 7.3.6 *Let (a_n) diverge to infinity and (b_n) be bounded. Then, $a_n + b_n \to \infty$.*

Proof: Let $K \in \mathbb{R}$ be given. Since (b_n) is bounded, there exists $M \in \mathbb{N}$ such that $|b_n| \leq M$ (in particular, $b_n \geq -M$) for all n. Since $a_n \to \infty$, there exists $N \in \mathbb{N}$ such that $a_n > K + M$ for all $n > N$. Therefore, $a_n + b_n > K + M - M = K$ whenever $n > N$, as needed. □

Proposition 7.3.7 *Let (a_n) diverge to infinity but $a_n \neq 0$ for all $n \in \mathbb{N}$. Then, $1/a_n \to 0$.*

Proof: Let $\varepsilon > 0$ be given. Since $a_n \to \infty$, there exists $N \in \mathbb{N}$ such that $a_n > 1/\varepsilon$ for all $n > N$. For $n > N$, this implies that $|1/a_n| = 1/a_n < \varepsilon$, as required. □

Definition 7.3.8 Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$.

(i) If D is unbounded **above**, f has a limit $L \in \mathbb{R}$ at ∞ if, for any sequence (x_n) in D with $x_n \to \infty$, we have $f(x_n) \to L$. This is denoted $\lim_{x\to\infty} f(x) = L$.

(ii) If D is unbounded **below**, f has a limit $L \in \mathbb{R}$ at $-\infty$ if, for any sequence (x_n) in D with $x_n \to -\infty$, we have $f(x_n) \to L$. This is denoted $\lim_{x\to-\infty} f(x) = L$.

Theorem 7.3.9 (ε-K Limits at Infinity) *Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$ is unbounded above. Then, $\lim_{x\to\infty} f(x) = L$ if and only if for each $\varepsilon > 0$, there exists $K \in \mathbb{R}$ such that, for all $x \in D$ with $x > K$, we have $|f(x) - L| < \varepsilon$.*

Sketch of Proof: This is very similar to the proofs of Theorem 5.1.3 and Theorem 7.2.1. □

Theorem 7.3.12 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with limits at both $\pm\infty$. Then, f is bounded.*

Proof: Let $L := \lim_{x\to\infty} f(x)$ and $K := \lim_{x\to-\infty} f(x)$. By the ε-K property of limits at infinity, there exists $R_1 > 0$ such that $|f(x) - L| < 1$ (that is $f(x) \leq |L| + 1$) for all $x > R_1$. Similarly, there exists $R_2 > 0$ such that $|f(x) - K| < 1$ (that is $f(x) \leq |K| + 1$) for all $x < -R_2$. Because f is continuous on the interval $[-R_2, R_1]$, it is bounded there by some N. Thus, $|f(x)| \leq M$ for all $x \in \mathbb{R}$, where $M := \max\{|L| + 1, |K| + 1, N\}$. □

8 Complex Sequences and Series

8.1 Convergence of Complex Sequences

Note: Recall that a complex number $z \in \mathbb{C}$ is a number of the form $z = x + iy$, where $x, y \in \mathbb{R}$ are the real part and imaginary part of z, respectively; these are $\text{Re}(z)$ and $\text{Im}(z)$. The modulus of $z = x + iy$ is the real number defined as $|z| = \sqrt{x^2 + y^2}$. The complex conjugate is the complex number defined as $\bar{z} = x - iy$.

Proposition 8.1.1 *Let $z, w \in \mathbb{C}$. Then, the following are true:*

(i) $\text{Re}(z) \leq |z|$.

(ii) $\text{Im}(z) \leq |z|$.

(iii) $|zw| = |z||w|$. (**Multiplicativity**)

(iv) $|z + w| \leq |z| + |w|$. (**Triangle Inequality**)

Definition A sequence of complex numbers is a function $z : \mathbb{N} \to \mathbb{C}$ where we call the output z_n. We denote a term in the sequence by z_n and the whole sequence by $(z_n)_{n \in \mathbb{N}}$, or (z_n).

Definition 8.1.2 A complex sequence (z_n) converges to $L \in \mathbb{C}$ if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $|a_n - L| < \varepsilon$. Here, we write $z_n \to L$.

Remark We explain Definition 8.1.2 in words and with a geometric interpretation in Figure 4.

(i) Given a complex sequence (z_n), we can show that it converges to $L \in \mathbb{C}$ by showing that for **any** positive number ($\varepsilon > 0$), there exists a point in the sequence a_N (there exists $N \in \mathbb{N}$) after which (for all $n > N$) every term in the sequence lies within distance that positive number of the complex number L ($|z_n - L| < \varepsilon$). This is very much the same idea as that for real sequences. The difference is the geometric interpretation

(ii) Geometrically, this means that if we plot the outputs z_n in the complex plane, then after N, every point will live **inside** the disk of radius ε centred at the point L.

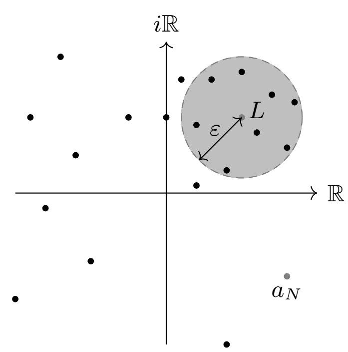

Figure 4: The geometric interpretation of the convergence of the complex sequence (z_n).

Proposition 8.1.4 *Let (z_n) be a complex sequence where $z_n = x_n + iy_n$ for each $n \in \mathbb{N}$ and let $L = A + iB$ where $A, B \in \mathbb{R}$. Then, the following are equivalent:*

(i) $z_n \to L$.

(ii) $x_n \to A$ *and* $y_n \to B$ *in the usual sense.*

(iii) $|z_n - L| \to 0$.

Proof: ((i) $\Rightarrow$ (ii)) Let $\varepsilon > 0$ be given. As $z_n \to L$, there exists $N \in \mathbb{N}$ such that $|z_n - L| < \varepsilon$ for all $n > N$. That being said, we can see that

$$\begin{aligned}|z_n - L| &= |x_n + iy_n - (A + iB)| \\ &= |(x_n - A) + i(y_n - B)| \\ &= \sqrt{(x_n - A)^2 + (y_n - B)^2} \\ &\geq \max\{\sqrt{(x_n - A)^2}, \sqrt{(y_n - B)^2}\} \\ &= \max\{|x_n - A|, |y_n - B|\}.\end{aligned}$$

Therefore, $|x_n - A| \leq |z_n - L| < \varepsilon$ and $|y_n - B| \leq |z_n - L| < \varepsilon$ for all $n > N$. This is precisely to say that the real sequences $x_n \to A$ and $y_n \to B$.

((ii) $\Rightarrow$ (iii)) Let $\varepsilon > 0$ be given. Since $x_n \to A$, there exists $N_1 \in \mathbb{N}$ such that, for all $n > N_1$, $|x_n - A|\varepsilon/2$. Similarly, as $y_n \to B$, there exists $N_2 \in \mathbb{N}$ such that, for all $n > N_2$, $|y_n - B| < \varepsilon/2$. Hence, for $N := \max\{N_1, N_2\}$ and $n > N$,

$$\begin{aligned}||z_n - L| - 0| &= |z_n - L| \\ &= |(x_n - A) + i(y_n - B)| \\ &\leq |x_n - A| + |y_n - B| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon.\end{aligned}$$

This is precisely to say that $|z_n - L| \to 0$.

((iii) $\Rightarrow$ (i)) Let $\varepsilon > 0$ be given. Since $|z_n - L| \to 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $||z_n - L| - 0| < \varepsilon$. But this is just $|z_n - L| < \varepsilon$ for all $n > N$, which is precisely to say that $z_n \to L$. □

Note: Here are some important consequences of Proposition 8.1.4:

- The limit of a convergent complex sequence is unique.
- Every convergent complex sequence is bounded.
- The Algebra of Limits holds for convergent complex sequences.

Definition 8.1.8 A function $f : \mathbb{C} \to \mathbb{C}$ is continuous at $w \in \mathbb{C}$ if, for **all** sequences (z_n) in $\mathbb{C}$ where $z_n \to w$, we have $f(z_n) \to f(w)$. If f is **not** continuous at $w \in \mathbb{C}$, we say it is discontinuous at $w \in \mathbb{C}$. We say f is continuous if it is continuous at **every** $w \in \mathbb{C}$.

Proposition 8.1.9 *Let $f : \mathbb{C} \to \mathbb{C}$ be continuous and (z_n) be a sequence defined inductively via iterating f from a starting value $z_0 \in \mathbb{C}$, that is $z_n = f(z_{n-1})$. If $z_n \to L$, then L is a* fixed point *of f, meaning $f(L) = L$.*

Proof: Let $z_n \to L$. By Proposition 8.1.4 combined with Theorem 3.1.3, we know that the subsequence $z_{n+1} \to L$ also. By definition, $z_{n+1} = f(z_n)$. By continuity, $f(z_n) \to f(L)$, so the Uniqueness of Limits guarantees that $f(L) = L$. □

8.2 Complex Series

Definition 8.2.1 For a complex series $\sum_{n=1}^{\infty} z_n$, its k^{th} partial sum is $s_k := \sum_{n=1}^{k} z_n$. We say that the series converges if the sequence of partial sums (s_k) converges in the usual sense (of Definition 8.1.2). We say that the series converges absolutely if the **real** series $\sum_{n=1}^{\infty} |z_n|$ converges in the usual sense.

Note: By Proposition 8.1.4, if $z_n = x_n + iy_n$ for all $n \in \mathbb{N}$, the complex series $\sum_{n=1}^{\infty} z_n$ converges if and only if both the **real** series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge.

Proposition 8.2.3 *If a complex series $\sum_{n=1}^{\infty} z_n$ converges absolutely, then it converges.*

Proof: Let $z_n = x_n + iy_n$ and assume $\sum_{n=1}^{\infty} z_n$ converges absolutely, i.e. the real sequence (s_k) converges where $s_k = \sum_{n=1}^{k} |z_n|$. Consequently, (s_k) is bounded above (note that the sequence of partial sums is increasing). Since $|x_n| \le |z_n|$ and $|y_n| \le |z_n|$ by Proposition 8.1.1, consider these:

$$u_k := \sum_{n=1}^{k} |x_n| \le s_k \quad \text{and} \quad v_k := \sum_{n=1}^{k} |y_n| \le s_k.$$

Because (s_k) is bounded above, so too are the sequences (u_k) and (v_k). Moreover, each of these sequences is increasing, so the MCT implies that they converge. Because absolute convergence implies convergence of **real** series (Theorem 4.4.2), we know that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge. Thus, the series $\sum_{n=1}^{\infty} z_n$ converges by Proposition 8.1.4. □

8.3 Power Series

Definition A power series is a complex series of the form $\sum_{n=0}^{\infty} a_n z^n$ where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$ (we switch notation from z_n denoting complex series because $z_n z^n$ looks rather confusing!). Here, z is a complex **variable**, so the k^{th} partial sum is the following polynomial:

$$s_k = \sum_{n=0}^{k} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k.$$

Theorem *The series $\sum_{n=1}^{\infty} z^n$ converges for $|z| < 1$ and diverges otherwise.*

Proof: As in the proof of the convergence of the geometric series, the k^{th} partial sum s_k is

$$\begin{aligned} & s_k = 1 + z + z^2 + \cdots + z^k \\ \Rightarrow \quad & zs_k = z + z^2 + z^3 + \cdots + z^{k+1} \\ \Rightarrow \quad & (1-z)s_k = 1 - z^{k+1} \\ \Rightarrow \quad & s_k = \frac{1 - z^{k+1}}{1-z}. \end{aligned}$$

If $|z| < 1$, then $|z|^{k+1} \to 0$ which means that $s_k \to \frac{1}{1-z}$. Hence, we know that the series converges for $|z| < 1$. On the other hand, suppose now that $|z| \geq 1$; this means that each term $|z^n| \geq 1$. Since they are complex numbers, we can write $z^n = x_n + iy_n$ for all $n \in \mathbb{N}$. Thus, the previous inequality is equivalent to $x_n^2 + y_n^2 \geq 1$. We then see that either $x_n \not\to 0$ or $y_n \not\to 0$, which implies that either $\sum_{n=1}^{\infty} x_n$ diverges or $\sum_{n=1}^{\infty} y_n$ diverges, by the Divergence Test. Hence, Proposition 8.1.4 implies that $\sum_{n=1}^{\infty} z^n$ diverges for all $|z| \geq 1$. □

Note: If $z = 0$, the power series reduces to the constant a_0, which means it certainly converges. In general, a power series may converge for some values of z but not others.

Definition 8.3.2 The exponential, sine and cosine functions are defined as follows:

$$\begin{aligned} \exp : \mathbb{C} \to \mathbb{C}, \qquad & \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\ \sin : \mathbb{C} \to \mathbb{C}, \qquad & \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \\ \cos : \mathbb{C} \to \mathbb{C}, \qquad & \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}. \end{aligned}$$

Proposition *The exponential function is well-defined: $\sum_{n=0}^{\infty} z^n/n!$ converges for all $z \in \mathbb{C}$.*

Proof: We know that it converges at $z = 0$ already, by the above note. Hence, assume that $z \neq 0$ and let $w_n = z^n/n!$. It is clear that $|w_n| > 0$ and

$$\frac{|w_{n+1}|}{|w_n|} = \frac{|z|^{n+1}}{(n+1)!} \frac{n!}{|z|^n} = \frac{|z|}{n+1} \to 0 < 1.$$

So, $\sum_{n=1}^{\infty} |w_n|$ converges by the Ratio Test, and $\sum_{n=1}^{\infty} w_n$ converges by Proposition 8.2.3. □

Definition 8.3.5 The radius of convergence of a power series $\sum_{n=0}^{\infty} a_n z^n$ is the constant

$$R \coloneqq \sup\{|z| : \sum_{n=0}^{\infty} |a_n z^n| \text{ converges}\}.$$

If the set on the right-hand side is unbounded, the convention is to write $R = \infty$.

Note: In other words, the radius of convergence is the positive constant $R \geq 0$ such that the power series converges for all values of $z \in \mathbb{C}$ within that distance from the origin (that is $|z| < R$) and diverges for all values more than that distance away from the origin (that is $|z| > R$). The power series may **converge or diverge** for $z \in \mathbb{C}$ where $|z| = R$.

Lemma 8.3.6 *If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = w$, then it converges* **absolutely** *for $|z| < |w|$.*

Proof: As $\sum_{n=0}^{\infty} a_n w^n$ converges, so do the series of real and imaginary parts (Proposition 8.1.4). By the Divergence Test, it follows that the **real** sequences $(\mathrm{Re}(a_n w^n))$ and $(\mathrm{Im}(a_n w^n))$ converge to zero. Therefore, $a_n w^n \to 0$ again by Proposition 8.1.4. In particular, this sequence is bounded, i.e. there exists $K > 0$ such that, for all $n \in \mathbb{N}$, $|a_n w^n| < K$. But we see that

$$|a_n z^n| = |a_n w^n| \frac{|z^n|}{|w^n|} < K \left(\frac{|z|}{|w|} \right)^n.$$

Since $|z| < |w|$, we can use the Comparison Test to conclude $\sum_{n=0}^{\infty} |a_n z^n|$, and therein $\sum_{n=0}^{\infty} a_n z^n$, converges for all $|z| < |w|$; we compare this to the convergent geometric series $\sum_{n=0}^{\infty} (|z|/|w|)^n$. □

Theorem 8.3.7 *Let $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. There are two possibilities:*

(i) *The power series converges* **absolutely** *for $|z| < R$.*

(ii) *The power series diverges for $|z| > R$.*

Proof: Define the set $A \coloneqq \{|z| : \sum_{n=0}^{\infty} |a_n z^n| \text{ converges}\} \subseteq \mathbb{R}$; this means that $R = \sup(A)$.

(i) If $|z| < R$, then there exists $w \in \mathbb{C}$ with $|z| < |w| < R$ such that $\sum_{n=0}^{\infty} |a_n w^n|$ converges (we know this is true because, if not, R wouldn't be the **least** upper bound on A). Thus, $\sum_{n=0}^{\infty} a_n w^n$ is convergent, and so $\sum_{n=0}^{\infty} a_n z^n$ is **absolutely** convergent by Lemma 8.3.6.

(ii) If $|z| > R$, assume to the contrary that $\sum_{n=0}^{\infty} a_n z^n$ converges. This implies $\sum_{n=0}^{\infty} |a_n v^n|$ converges where $v = \frac{1}{2}(|z| + R)$; this is another application of Lemma 8.3.6 since $|z| < |v|$. But $|v| > R$ (so R is **not** an upper bound on A), a contradiction. □

Note: We give a method for computing the radius of convergence via the Ratio Test. That being said, be aware that it will **not always** work. Furthermore, note that Definition 8.3.5 has **no** mention of the Ratio Test.

Method – Finding the Radius of Convergence: Consider the power series $\sum_{n=0}^{\infty} a_n z^n$.

(i) Let $b_n := a_n z^n$ be the sequence of terms in the series.

(ii) Compute $|b_{n+1}|/|b_n|$ and determine its limit as $n \to \infty$ in terms of $|z|$.

(iii) Use the Ratio Test, in particular that $|b_{n+1}|/|b_n| \to L < 1$ means the power series converges (absolutely) and $|b_{n+1}|/|b_n| \to L > 1$ means the power series diverges, to determine an upper bound on $|z|$ to make the limit in Step (ii) less than one.