MATH3001: Hyperbolic Geometry

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Abstract

Geometry has been well-understood for many years and the famous mathematicians of old have left their imprint on this topic. Arguably one of the most profound to do so was Carl Friedrich Gauss; he started to consider an altering of the basic axioms of what we now call Euclidean geometry. This paper will introduce one of these aforementioned alterations we call hyperbolic geometry, where we discuss it in the sense of the Poincaré disk. From here, we will develop theory on lines, transformations, lengths and areas by referring both to our definitions and the area of complex analysis. We shall then divert the discussion to focus on right-angled polygons in the hyperbolic plane, up to the study of pentagons.

Acknowledgements

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Contents

1 Metric Spaces, Isometries and Groups

To begin the process of defining a variety of hyperbolic geometric-related objects, first the idea of metric spaces shall be discussed in order to both simplify notation and interlink the concept of the hyperbolic plane with topological spaces. Then, the idea of an isometry shall be developed, before Section 3 leads on to a more specific idea, underpinned by the theory of isometries of hyperbolic geometry. Also, some group theory is required when discussing these isometries and so this will also be discussed briefly as well.

1.1 Introducing Metric Spaces

Definition 1.1 A metric space (X, d) is a non-empty set X and function $d : X \times X \to \mathbb{R}$, called the metric, which satisfies the following criteria:

- $d(x, y) \geq 0$, with equality if and only if $x = y$. (Non-Negativity)
- $d(x, y) = d(y, x)$. (Symmetry)
- $d(x, y) \leq d(x, z) + d(z, y)$. (Triangle Inequality)

Example 1.2 The set $X = \mathbb{R}^n$ with the function $d(x, y) = |x - y|$, for $x, y \in \mathbb{R}^n$, is an example of a metric space, where d is the standard Euclidean metric. This is because X is certainly non-empty and d satisfies the three conditions of Definition 1.1. Note that this is not the only metric which can be defined on X.

More can be seen on this topic in [DE14], which covers complex-valued functions, something that will connect to future areas of discussion. For now, we detail the construction of what is called the Riemannian metric on $\mathbb{R}^2$. One can see from [Cha13] that, locally in $\mathbb{R}^2$, we see that

$$(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2,$$

a consequence of Pythagoras' Theorem, where δs denotes an infinitesimal change of the length of a curve between two points, which is parametrised by the variable t say, and $(\delta x, \delta y)$ denotes an infinitesimal change in a point on the curve. Formally, as the change in parameter $\delta t \to 0$,

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2.$$

This is the so-called Riemannian metric. It is now possible to consider integrals of the metric as expressions of the distance between two points, as will be done in Section 3. Furthermore, we require a final property of metric spaces, which will be applied in the context of differentiable functions defined over the complex numbers.

Definition 1.3 Let (X, d) be a metric space. An open ball of radius $\varepsilon > 0$ and centre $x \in X$ is defined to be the set $B_\varepsilon(x) \coloneqq \{y \in X : d(x, y) < \varepsilon\} \subseteq X$.

Definition 1.4 Let (X, d) be a metric space and $U \subseteq X$. Then, U is said to be an open subset if, for all $x \in U$, there exists an open ball $B_\varepsilon(x)$ such that $B_\varepsilon(x) \subseteq U$.

1.2 Introducing Isometries

Definition 1.5 An isometry of a metric space (X, d) is a map $f : X \to X$ such that it preserves the metric, that is $d(f(x), f(y)) = d(x, y)$, for all $x, y \in X$.

The concept of an isometry will be useful for the study of both distance and transformations in the hyperbolic plane. But first, we consider a simple Euclidean example and discuss one way in which to represent linear isometries.

Example 1.6 A translation $T : \mathbb{R}^n \to \mathbb{R}^n$, given by $T(x) = x+v$ for a fixed $v \in \mathbb{R}^n$, is an example of an isometry, seen by applying the definition to the standard Euclidean metric:

$$d(T(x), T(y)) = |T(x) - T(y)| = |(x + v) - (y + v)| = |x - y| = d(x, y).$$

Proposition 1.7 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear isometry. Then, for $\theta \in [0, 2\pi)$, its matrix is either*

$$A_f = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \text{or} \qquad A_f = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix},$$

for f orientation-preserving $(\det(A_f) > 0)$ or orientation-reversing $(\det(A_f) < 0)$, respectively.

Proof: Suppose that the matrix of f is given by

$$A_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is to say $f(x, y) = (ax + by, cx + dy)$. Since f is an isometry, for the standard Euclidean metric, it must follow that

$$\begin{aligned} & d(f(x, y), f(u, v)) = d((x, y), (u, v)) \\ \Rightarrow \quad & |(ax + by, cx + dy) - (au + bv, cu + dv)| = |(x - u, y - v)| \\ \Rightarrow \quad & (a(x - u) + b(y - v))^2 + (c(x - u) + d(y - v))^2 = (x - u)^2 + (y - v)^2. \end{aligned}$$

Expanding the left-hand side yields the following equations:

$$a^2 + c^2 = 1, \tag{1.8}$$

$$ab + cd = 0, \tag{1.9}$$

$$b^2 + d^2 = 1. \tag{1.10}$$

By (1.8), we can set $a = \cos(\theta)$ and $c = \sin(\theta)$, for some $\theta \in [0, 2\pi)$. Thus, by (1.9), we find that $b = -\alpha \sin(\theta)$ and $d = \alpha \cos(\theta)$, for any $\alpha \in \mathbb{R}$. Substituting these into (1.10) gives $\alpha = \pm 1$, i.e. the cases of orientation-preserving and orientation-reversing, respectively. $\square$

Remark 1.11 The matrices in Proposition 1.7 give two situations: either where a vector in $\mathbb{R}^2$ is rotated or where it is rotated and reflected. This plants the seed of the discussion regarding what isometries can do in hyperbolic space.

1.3 Introducing Group Theory

Group theory will be applied to demonstrate useful properties of the transformations that can be performed in the hyperbolic plane. For now, consider these rudimentary concepts.

Definition 1.12 A binary operation $\circ$ on a set G is a map $\circ : G \times G \to G$.

Definition 1.13 A group is a set and binary operation, denoted $(G, \circ)$, satisfying the following:

(i) For all $x, y \in G,\ x \circ y \in G$.

(ii) For all $x, y, z \in G,\ x \circ (y \circ z) = (x \circ y) \circ z$.

(iii) There exists $e \in G$ such that $x \circ e = e \circ x = x$, for every $x \in G$.

(iv) For all $x \in G$, there exists some $y \in G$ such that $x \circ y = y \circ x = e$.

Example 1.14 Consider the following sets and operations:

(i) $(\mathbb{R}, +)$ is a group.

(ii) $(\mathbb{R}, \times)$ is **not** a group, since $0 \in \mathbb{R}$ has no multiplicative inverse.

Definition 1.15 A subgroup H of a group G is a subset $H \subseteq G$ such that $(H, \circ)$ is a group under the same operation as G, denoted $H \leq G$.

Example 1.16 Consider the following:

(i) The trivial subgroups of a group G are $\{e\}$ and G, where $e \in G$ is the identity of G.

(ii) $(\mathbb{Z}_2, +_{\text{mod } 2})$ is **not** a subgroup of $(\mathbb{Z}_3, +_{\text{mod } 3})$, since the group operation is different.

Theorem 1.17 (Subgroup Criterion) *Let $(G, \circ)$ be a group. $H \subseteq G$ is a subgroup if and only if the following criterion hold:*

(i) *For all $x, y \in H,\ x \circ y \in H$.*

(ii) *The identity $e \in G$ is also contained in H, that is $e \in H$.*

(iii) *For all $x \in H$, there exists some $y \in H$ such that $x \circ y = y \circ x = e$.*

Proof: See [Wal01]; the proof is not immediately required for this paper's discussion. $\square$

Definition 1.18 The general linear group is the set of invertible $(n \times n)$-matrices whose entries are in a given set under matrix multiplication, denoted $\mathrm{GL}_n(X)$, where X is the set of entries.

Definition 1.19 A group homomorphism between $(G, \circ_G)$ and $(H, \circ_H)$ is a map $\varphi : G \to H$ such that $\varphi(u \circ_G v) = \varphi(u) \circ_H \varphi(v)$, for all $u, v \in G$. If φ is additionally bijective, then it is called a group isomorphism, that is G and H are isomorphic, denoted $G \cong H$.

Example 1.20 The map $\exp : (\mathbb{R}, +) \to (\mathbb{R} \setminus \{0\}, \times)$ is a group homomorphism: for $x, y \in \mathbb{R}$,

$$\exp(x + y) = \exp(x) \exp(y).$$

Note this isn't a group *isomorphism* because exp doesn't map to the negative real numbers.

2 Models of Hyperbolic Geometry

There are a number of models one can use to describe hyperbolic geometry. For now, we restrict to two such, as defined below. Note that these models are subsets of $\mathbb{R}^2$ but that it is useful to associate with the point (x, y) the complex number $z = x + iy$.

Definition 2.1 The Poincaré disk and its boundary are defined respectively as

$$\mathbb{D} \coloneqq \{z \in \mathbb{C} : |z| < 1\} \qquad \text{and} \qquad \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}.$$

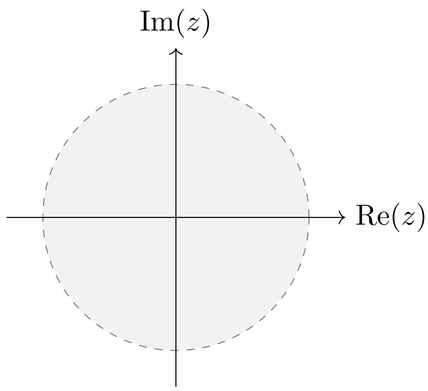

Figure 1: The Poincaré disk, as in Definition 2.1.

The elements of the set $\mathbb{D}$ are indeed the points of the model of hyperbolic geometry discussed in the above definition. However, the elements of the set $\partial\mathbb{D}$ are **not** points of the model. In fact, Section 3 and, to an extent, Section 4 introduce the notion of hyperbolic length/distance and with this, the points of $\partial\mathbb{D}$ are found to be infinitely far from the points of $\mathbb{D}$.

Definition 2.2 The Poincaré upper-half plane and its boundary are defined respectively as

$$\mathbb{H} \coloneqq \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \qquad \text{and} \qquad \partial\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\} \cup \{\infty\}.$$

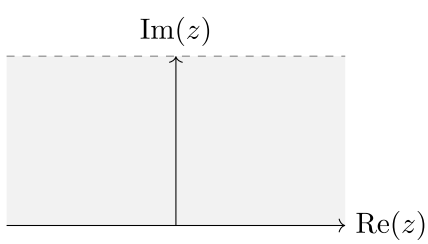

Figure 2: The Poincaré upper-half plane, as in Definition 2.2.

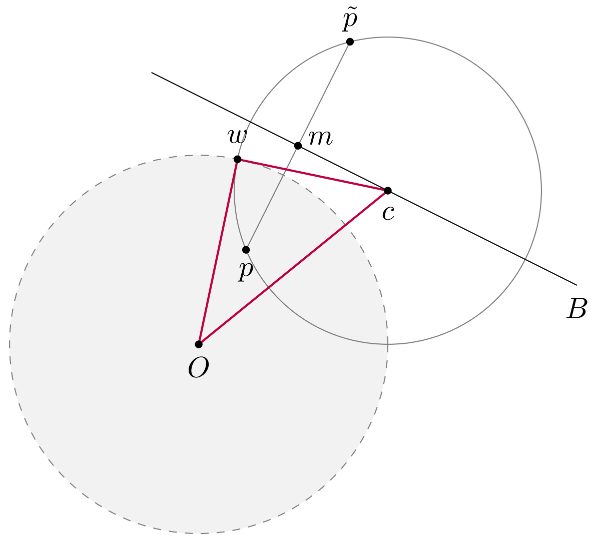

Figure 3: A hyperbolic line meets the Poincaré disk's boundary, as in Lemma 2.7.

For this paper, we restrict solely to the disk model as in Definition 2.1. This is, in a way, advantageous for visualising the geometry in this introduction since it isn't a set which extends indefinitely as the upper-half plane does (the union of the disk and its boundary is compact).

2.1 Lines of the Poincaré Disk

Definition 2.3 A hyperbolic line L in the Poincaré disk is one of the following:

(i) The intersection of a Euclidean line through the origin and $\mathbb{D}$.

(ii) The intersection of a Euclidean circle and $\mathbb{D}$, which meets $\partial\mathbb{D}$ at right angles.

We now develop theory which can be exploited to construct such lines.

Lemma 2.4 *Consider two Euclidean circles $C_1, C_2 \subseteq \mathbb{C}$ with centres $c_1, c_2 \in \mathbb{C}$ and radii $r_1, r_2 > 0$, respectively. Then, C_1 and C_2 meet at right angles if and only if $r_1^2 + r_2^2 = |c_1 - c_2|^2$. In particular, for C_1 any circle and C_2 the unit circle, this defines the condition on which a line can be as in* Definition 2.3.

Proof: We find relationship between two Euclidean circles in [BEG12], namely for the circles

$$x^2 + y^2 + f_1x + g_1y + h_1 = 0 \qquad \text{and} \qquad x^2 + y^2 + f_2x + g_2y + h_2 = 0,$$

they are orthogonal if and only if

$$f_1f_2 + g_1g_2 = 2(h_1 + h_2). \tag{2.5}$$

Suppose C_1 has radius r_1, centre $c_1 = (a_1, b_1)$ and C_2 has radius r_2, centre $c_2 = (a_2, b_2)$. The functions above can be determined by comparing them to the equations of C_1 and C_2:

$$x^2 + y^2 + \underbrace{(-2a_1)}_{f_1} x + \underbrace{(-2b_1)}_{g_1} y + \underbrace{a_1^2 + b_1^2 - r_1^2}_{h_1} = 0$$

and

$$x^2 + y^2 + \underbrace{(-2a_2)}_{f_2} x + \underbrace{(-2b_2)}_{g_2} y + \underbrace{a_2^2 + b_2^2 - r_2^2}_{h_1} = 0.$$

Then, the above expressions can be substituted into (2.5), leading to the intended result. □

Definition 2.6 Given some circle C in the complex plane, two points z and $\tilde{z}$ are said to be inverse points if any circle orthogonal to C containing z also contains $\tilde{z}$, where $\tilde{z} := z/|z|^2$.

Lemma 2.7 *Let $p \in \mathbb{D} \setminus \{0\}$. Any circle containing p and $\tilde{p}$ intersects $\partial\mathbb{D}$ at right-angles.*

Proof: Let $m \in \mathbb{C}$ be the midpoint of the Euclidean line with endpoints p and $\tilde{p}$ and B be its perpendicular bisector, where we use [Roy08] as a loose framework. Take c a point on the bisector B and define $r := \operatorname{dist}(p, c)$, as in Figure 3. Note as m is the midpoint of p and $\tilde{p}$, it follows that $|p - m| = \frac{1}{2}|\tilde{p} - p|$. Then, one can apply Pythagoras' Theorem, giving

$$\begin{aligned}
|c|^2 &= |m|^2 + |m - c|^2 \\
&= (|p| + |p - m|)^2 + |p - c|^2 - |p - m|^2 \\
&= \left(|p| + \frac{1}{2}\left(\frac{1}{|p|} - |p|\right)\right)^2 + r^2 - \frac{1}{4}\left(\frac{1}{|p|} - |p|\right)^2 \\
&= \frac{1}{4}\left(\frac{1}{|p|} + |p|\right)^2 + r^2 - \frac{1}{4}\left(\frac{1}{|p|} - |p|\right)^2 \\
&= r^2 + 1.
\end{aligned}$$

Consequently, Lemma 2.4 applies to $\partial\mathbb{D}$ and the circle of radius r, centre c. □

Proposition 2.8 *There exists a unique hyperbolic line in the Poincaré disk such that it contains any two distinct points $p, q \in \mathbb{D}$.*

Proof: There are two cases to consider.

(i) p and q lie on a diameter of $\mathbb{D}$: Let L be the intersection of a Euclidean line with the disk $\mathbb{D}$ which contains p and q. Then, L is a diameter of $\mathbb{D}$, that is it passes through the origin. Hence, it is a hyperbolic line.

Suppose that another such line K is such that it contains p and q. Then, it is either the intersection with the disk $\mathbb{D}$ of a Euclidean line or a Euclidean circle. Suppose it is the latter. Then, the perpendicular bisectors B_1 of the Euclidean line joining p and q and B_2 of the Euclidean line joining p and $\tilde{p}$ are parallel, meaning no such circular arc contains p and q, a contradiction. Thus, K must be a diameter of $\mathbb{D}$ containing p and q. Hence, K must be L.

(ii) p and q don't lie on a diameter of $\mathbb{D}$: Let L be the intersection of a Euclidean circle with the disk $\mathbb{D}$ which contains p and q, with radius $r_L > 0$ and centre $c_L \in \mathbb{C}$, which is the point of intersection of the perpendicular bisectors B_1 of the Euclidean line joining p and q and B_2 of the Euclidean line joining p and $\tilde{p}$. Then, L meet $\partial\mathbb{D}$ at right-angles, by Lemma 2.7. Hence, L is a hyperbolic line.

Suppose that another such line K is such that it contains p and q. Then, it is either the intersection with the disk $\mathbb{D}$ of a Euclidean line or a Euclidean circle. Suppose it is the former. Then, K does not pass through the origin as p and q aren't on a diameter of $\mathbb{D}$, so it isn't a hyperbolic line, a contradiction. Hence, it must be the arc of a circle, whose radius is $r_K > 0$ and centre is $c_K \in \mathbb{C}$. But then, c_K must lie on B_1 and B_2, ensuring equidistance between p and q and perpendicularity with $\partial\mathbb{D}$. Therefore, $c_K = c_L$. As a consequence, $r_K = \operatorname{dist}(c_K, p) = \operatorname{dist}(c_L, p) = r_L$. Hence, K must be L.

Thus, in any case, a unique hyperbolic lines exist between two distinct points. □

Proposition 2.9 *There exists a unique hyperbolic line in the Poincaré disk with ends on any two distinct points $p, q \in \partial\mathbb{D}$.*

Sketch of Proof: Let $p, q \in \partial\mathbb{D}$. As with the proof of Proposition 2.8, there are two cases to consider; either the points lie on a diameter of $\mathbb{D}$ or they do not. If they do, then such a diameter is unique and has endpoints p and q, by the same line of reasoning as the first case in the previous proof. Otherwise, if p and q are not on a diameter of $\mathbb{D}$, then the construction of the arc of the Euclidean circle as laid out in Proposition 2.8 is such that $\tilde{p} = p$, since $|p| = 1$. Thus, the bisector B_2 is a tangent to $\partial\mathbb{D}$ at p and the construction of the Euclidean circle which intersects $\partial\mathbb{D}$ at right-angles proceeds as per Proposition 2.8. Again, uniqueness is achieved by a similar argument of the second case in the previous proof. □

2.2 Introducing Möbius Transformations

Transformations are an important part of geometry; the case of hyperbolic geometry is no exception. We now begin to introduce the idea which will be developed until Section 4.

Definition 2.10 The Riemann sphere is $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e. the complex plane with the point at infinity.

Definition 2.11 A Möbius transformation is a self-map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the form

$$f(z) = \frac{az+b}{cz+d},$$

for $a, b, c, d \in \mathbb{C}$ where $ad - bc \neq 0$. The set of such Möbius transformations is denoted $\mathcal{M}(\overline{\mathbb{C}})$.

Remark 2.12 Note that the function in Definition 2.11 maps from and to the Riemann sphere bijectively, as discussed in [Law98], so the point at infinity both has an image and a pre-image. Naturally, the image of this point is defined as a/c and the pre-image defined as $-d/c$. These points are often referred to as the inverse pole and the pole, respectively.

Möbius transformations are classified as orientation-preserving and orientation-reversing, whose sets are denoted $\mathcal{M}(\overline{\mathbb{C}})^+$ and $\mathcal{M}(\overline{\mathbb{C}})^-$, respectively. Such a classification will be discussed more specifically in Proposition 2.14. Not all Möbius transformations map from and to the Poincaré disk, motivating the following definition.

Definition 2.13 The set of Möbius transformations on the Poincaré disk, denoted $\mathcal{M}(\mathbb{D})$, is a subset of $\mathcal{M}(\overline{\mathbb{C}})$ that fixes both the disk $\mathbb{D}$ and the boundary $\partial\mathbb{D}$.

Proposition 2.14 *Let f be a Möbius transformation. It holds true that $f \in \mathcal{M}(\mathbb{D})$ if either*

$$f(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \tag{2.15}$$

or

$$f(z) = \frac{\alpha \bar{z} + \beta}{\bar{\beta} \bar{z} + \bar{\alpha}}, \tag{2.16}$$

where $\alpha, \beta \in \mathbb{C}$ are such that $|\alpha|^2 - |\beta|^2 > 0$.

Proof: Suppose f is of the form (2.15). Note it is sufficient to prove that $|f(z)| \leq 1$, with equality occurring when $|z| = 1$. For $|z| < 1$, this is equivalent to $|\alpha z + \beta| < \left|\bar{\beta} z + \bar{\alpha}\right|$. Consider these:

$$\begin{aligned}
|\alpha z + \beta|^2 &= |\alpha|^2 |z|^2 + |\beta|^2 + \alpha\bar{\beta} z + \bar{\alpha}\beta\bar{z}, \\
\left|\bar{\beta} z + \bar{\alpha}\right|^2 &= |\beta|^2 |z|^2 + |\alpha|^2 + \alpha\bar{\beta} z + \bar{\alpha}\beta\bar{z}.
\end{aligned}$$

By assumption, $|\alpha|^2 - |\beta|^2 > 0 \Rightarrow |\alpha|^2 > |\beta|^2$. In particular, this means that for some $\delta < 0$,

$$\delta|\alpha|^2 < \delta|\beta|^2.$$

Let $\delta := |z|^2 - 1 < 0$, since $|z| < 1$ by assumption. Then, the following is true for any $k \in \mathbb{R}$:

$$\begin{aligned} & |\alpha|^2(|z|^2-1) < |\beta|^2(|z|^2-1) \\ \Rightarrow \quad & |\alpha|^2|z|^2 - |\alpha|^2 < |\beta|^2|z|^2 - |\beta|^2 \\ \Rightarrow \quad & |\alpha|^2|z|^2 + |\beta|^2 < |\beta|^2|z|^2 + |\alpha|^2 \\ \Rightarrow \quad & |\alpha|^2|z|^2 + |\beta|^2 + k < |\beta|^2|z|^2 + |\alpha|^2 + k. \end{aligned}$$

Choosing $k = \alpha\bar{\beta}z + \bar{\alpha}\beta\bar{z}$ (which is real since this is the sum of a complex number and its conjugate, so the imaginary part vanishes) in the final line above implies $|\alpha z + \beta| < \left|\bar{\beta}z + \bar{\alpha}\right|$. Note that if $|z| = 1$, then $|\alpha z + \beta|^2 = \left|\bar{\beta}z + \bar{\alpha}\right|^2$, that is $|f(z)|^2 = 1$ so it fixes $\partial\mathbb{D}$. Hence, $f \in \mathcal{M}(\mathbb{D})$. A near-identical argument applies in the case that f is of the form (2.16). ☐

Example 2.17 The Möbius transformation $g : \mathbb{H} \to \mathbb{D}$, given by

$$g(z) = \frac{z-i}{z+i},$$

is such that it maps $1 \mapsto -i$, $0 \mapsto -1$ and $\infty \mapsto 1$, which means that $\mathbb{R} \mapsto \partial\mathbb{D}$. Thus, the Poincaré disk is the image of the Möbius transformation. This is one way to suggest that the models of hyperbolic geometry in Definition 2.1 and 2.2 are equivalent.

Remark 2.18 In fact, Proposition 2.14 is a necessary and sufficient condition on the form of the Möbius transformations of the Poincaré disk. In order to prove that such a transformation has one of the forms stated, one can appeal to results known about $\mathcal{M}(\mathbb{H})$, the Möbius transformations that fix the upper-half plane, as detailed in [And05].

2.3 Möbius Groups

Now that the Möbius transformations of the Poincaré disk have been classified, we can state and prove some important results regarding a subgroup of these isometries. First, we will state, but not prove, a simple result which will be used to prove another idea.

Lemma 2.19 $(\mathcal{M}(\overline{\mathbb{C}}), \circ)$ *forms a group, under function composition.*

Next, we define an object which makes calculations with Möbius transformations less tedious.

Definition 2.20 The matrix of a Möbius transformation $f(z) = (az+b)/(cz+b)$ is defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Remark 2.21 By Definition 2.11, the Möbius transformation being non-degenerate $(ad - bc \neq 0)$

corresponds to its matrix being invertible. Essentially, we have defined a group homomorphism $\varphi : \mathrm{GL}_2(\mathbb{C}) \to \mathcal{M}(\overline{\mathbb{C}})$, as is discussed in [Ols10].

Proposition 2.22 $(\mathcal{M}(\mathbb{D}), \circ) \leq (\mathcal{M}(\overline{\mathbb{C}}), \circ)$, *that is the group of Möbius transformations of the Poincaré disk forms a subgroup of all Möbius transformations, under function composition.*

Proof: We appeal to the Subgroup Criterion in order to prove the proposition. First, choose $f, g \in \mathcal{M}(\mathbb{D})$. For $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 > 0$, we define these Möbius transformations as

$$f(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \in \mathcal{M}(\mathbb{D})^+ \qquad \text{or} \qquad f(z) = \frac{\alpha \bar{z} + \beta}{\bar{\beta} \bar{z} + \bar{\alpha}} \in \mathcal{M}(\mathbb{D})^-$$

and

$$g(z) = \frac{\gamma z + \delta}{\bar{\delta} z + \bar{\gamma}} \in \mathcal{M}(\mathbb{D})^+ \qquad \text{or} \qquad g(z) = \frac{\gamma \bar{z} + \delta}{\bar{\delta} \bar{z} + \bar{\gamma}} \in \mathcal{M}(\mathbb{D})^-.$$

Hence, we quickly compute the following compositions:

$$(f \circ g)(z) = \begin{cases} \dfrac{(\alpha\gamma + \beta\bar{\delta})z + (\alpha\delta + \beta\bar{\gamma})}{(\bar{\alpha}\bar{\delta} + \bar{\beta}\gamma)z + (\bar{\alpha}\bar{\gamma} + \bar{\beta}\delta)} & \text{if } f \in \mathcal{M}(\mathbb{D})^+ \text{ and } g \in \mathcal{M}(\mathbb{D})^+ \\[2ex] \dfrac{(\alpha\gamma + \beta\bar{\delta})\bar{z} + (\alpha\delta + \beta\bar{\gamma})}{(\bar{\alpha}\bar{\delta} + \bar{\beta}\gamma)\bar{z} + (\bar{\alpha}\bar{\gamma} + \bar{\beta}\delta)} & \text{if } f \in \mathcal{M}(\mathbb{D})^\pm \text{ and } g \in \mathcal{M}(\mathbb{D})^\mp \\[2ex] \dfrac{(\alpha\bar{\gamma} + \beta\delta)z + (\alpha\bar{\delta} + \beta\gamma)}{(\bar{\alpha}\delta + \bar{\beta}\bar{\gamma})z + (\bar{\alpha}\gamma + \bar{\beta}\bar{\delta})} & \text{if } f \in \mathcal{M}(\mathbb{D})^- \text{ and } g \in \mathcal{M}(\mathbb{D})^- \end{cases}.$$

The determinant of the transformation's matrix is $(|\alpha|^2 - |\beta|^2)(|\gamma|^2 - |\delta|^2) > 0$. Therefore, $f \circ g \in \mathcal{M}(\mathbb{D})$, demonstrating *composition closure*. Next, id $= (1z + 0)/(0z + 1)$ is such that $|1|^2 - |0|^2 = 1 > 0$, that is id $\in \mathcal{M}(\mathbb{D})$. As such, this is the *identity element*. Next, again choose $f \in \mathcal{M}(\mathbb{D})$. For $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 > 0$, we define the Möbius transformation as

$$f(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \in \mathcal{M}(\mathbb{D})^+ \qquad \text{or} \qquad f(z) = \frac{\alpha \bar{z} + \beta}{\bar{\beta} \bar{z} + \bar{\alpha}} \in \mathcal{M}(\mathbb{D})^-.$$

Therefore, we conclude that

$$f^{-1}(z) = \begin{cases} \dfrac{\bar{\alpha} z - \beta}{-\bar{\beta} z + \alpha} & \text{if } f \in \mathcal{M}(\mathbb{D})^+ \\[2ex] \dfrac{\bar{\alpha} \bar{z} - \beta}{-\bar{\beta} \bar{z} + \alpha} & \text{if } f \in \mathcal{M}(\mathbb{D})^- \end{cases}.$$

Again, the determinant of the matrix representing the transformation is always $|\alpha|^2 - |\beta|^2 > 0$. Therefore, $f^{-1} \in \mathcal{M}(\mathbb{D})$. Thus, *closure under forming inverses* is verified. As a consequence of Lemma 2.19, this result is a corollary of the Subgroup Criterion. $\square$

Corollary 2.23 $(\mathcal{M}(\mathbb{D})^+, \circ) \leq (\mathcal{M}(\overline{\mathbb{C}})^+, \circ)$*, that is the group of orientation-preserving Möbius transformations of the Poincaré disk is a subgroup of all orientation-preserving Möbius transformations, under function composition.*

Proof: This is a direct consequence of the proof of Proposition 2.22: composition closure comes from the first case of the composition of elements in $\mathcal{M}(\mathbb{D})$; inclusion of the identity and inverse closure comes from the first case of the inverses of elements in $\mathcal{M}(\mathbb{D})$, thus the Subgroup Criterion is satisfied. $\square$

These results are not so deep but are important; they tell us that properties of $\mathcal{M}(\overline{\mathbb{C}})$ are inherited by $\mathcal{M}(\mathbb{D})$, a fact needed for later discussion.

2.4 Möbius Transformations on Points and Lines

We now consider the action of Möbius transformations on subsets of the Poincaré disk, which will be important for the theory developed in Section 4.

Proposition 2.24 *The image under $f \in \mathcal{M}(\mathbb{D})^+$ of a hyperbolic line L is a hyperbolic line.*

Proof: As f is a continuous function which fixes $\mathbb{D}$, the image of $\partial\mathbb{D}$ is $\partial\mathbb{D}$. Now, the image of L under f is either going to be a Euclidean circle or a Euclidean line, by [Hit09]. Furthermore, $f(L)$ meets $\partial\mathbb{D}$ at right-angles, as Möbius transformations preserve angles (see Theorem 3.17) and L is already orthogonal to $\partial\mathbb{D}$, by assumption. Direct from the definition, L meets $\partial\mathbb{D}$ at right-angles if it's a circle or meets two tangents to $\partial\mathbb{D}$ at right-angles if it's a line. Hence, if $f(L)$ is a Euclidean circle, it must intersect $\partial\mathbb{D}$ at right-angles and if $f(L)$ is a Euclidean line, then it must be a diameter across $\mathbb{D}$. In either case, $f(L) \subseteq \mathbb{D}$ is a hyperbolic line. $\square$

Proposition 2.25 *For any points $p, q \in \mathbb{D}$, there exists $f \in \mathcal{M}(\mathbb{D})$ where $f(p) = q$.*

Proof: We wish to find some Möbius transformation $g \in \mathcal{M}(\mathbb{D})$ such that $g(p) = 0$, that is

$$g(p) = \frac{\alpha p + \beta}{\bar{\beta} p + \bar{\alpha}} = 0.$$

This is true if and only if the numerator is zero, i.e. $\alpha p + \beta = 0$ and $\bar{\beta} p + \bar{\alpha} = 1$, say. Solving these simultaneously gives that $\alpha = -1/(|p|^2 - 1)$ and $\beta = p/(|p|^2 - 1)$. Let h be a Möbius transformation where $h(q) = 0$ (of a form similar to g). By Proposition 2.22, $\mathcal{M}(\mathbb{D})$ is closed under both composition and forming inverses. Hence,

$$f := h^{-1} \circ g \in \mathcal{M}(\mathbb{D}).$$

By definition, $f : p \xmapsto{g} 0 \xmapsto{h^{-1}} q$, so the desired Möbius transformation is found. $\square$

Proposition 2.26 *For any hyperbolic lines $K, L \subseteq \mathbb{D}$, there exists $f \in \mathcal{M}(\mathbb{D})$ where $f(K) = L$.*

Sketch of Proof: For any points $p, q \in \partial\mathbb{D}$ there exists $g \in \mathcal{M}(\mathbb{D})$ such that $g(p) = 1$ and $g(q) = -1$. This provides the set-up for the proof. Now, by Proposition 2.9, there exists a unique hyperbolic line, K say, with ends on p and q. Thus, f transforms K to the real axis, since any Euclidean line is determined uniquely by two points. For a second line, L say, there exists some $h \in \mathcal{M}(\mathbb{D})$ such that $h(u) = 1$ and $h(v) = -1$, where $u, v \in \partial\mathbb{D}$ are the unique endpoints of the line, again via Proposition 2.9. By Proposition 2.22, $\mathcal{M}(\mathbb{D})$ is closed under both composition and forming inverses. Hence,

$$f \coloneqq h^{-1} \circ g \in \mathcal{M}(\mathbb{D}).$$

By definition, $f : K \xmapsto{g} \{\text{real axis}\} \xmapsto{h^{-1}} L$, so this is the desired Möbius transformation. $\square$

2.5 Classifying Möbius Transformations

We can now think of the *geometry* of a Möbius transformation. As with Euclidean geometry, a *transformation* is loaded in the sense that it covers a number of different situations, each determined by their fixed points. This is also true for Möbius transformations.

Definition 2.27 A fixed point of a map f is some element of the domain x where $f(x) = x$.

Definition 2.28 We make the following definitions of certain Möbius transformations.

(i) Call $f \in \mathcal{M}(\mathbb{D})^+$ a rotation if it fixes **one** point in $\mathbb{D}$.

(ii) Call $f \in \mathcal{M}(\mathbb{D})^+$ a horolation if it fixes **one** point on $\partial\mathbb{D}$.

(iii) Call $f \in \mathcal{M}(\mathbb{D})^+$ a translation if it fixes **two** points on $\partial\mathbb{D}$.

(iv) Call $f \in \mathcal{M}(\mathbb{D})^-$ a reflection if it fixes a **hyperbolic line** in $\mathbb{D}$.

(v) Call $f \in \mathcal{M}(\mathbb{D})^-$ a glide reflection if it fixes **two** points on $\partial\mathbb{D}$.

Example 2.29 Consider the following Möbius transformation: $f(z) = (2z + 1)/(z + 2)$. In order to compute the fixed points, assume that $f(w) = w$, for $w \in \mathbb{D}$. Therefore, we see that

$$\begin{aligned} 2w + 1 &= w^2 + 2w \\ \Rightarrow \qquad w^2 &= 1 \\ \Rightarrow \qquad w &= \pm 1. \end{aligned}$$

Hence, the fixed points of f are 1 and -1, so f is a translation, by Definition 2.28.

In practice, it can be a tedious process referring directly to the definition of a fixed point to find and classify those of a given Möbius transformation. So, assume that, for some given Möbius transformation, $f(w) = w$, for $w \in \overline{\mathbb{C}}$. Applying the formula of Definition 2.11 yields

$$\frac{aw+b}{cw+d} = w \quad \Rightarrow \quad cw^2 + (d-a)w - b = 0 \quad \Rightarrow \quad w = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4bc}}{2c}.$$

Particularly, if $g \in \mathcal{M}(\mathbb{D})^+$ and $g(w) = w$, for $w \in \mathbb{D}$, then the formula of Proposition 2.14 gives

$$\frac{\alpha w+\beta}{\bar{\beta} w+\bar{\alpha}} = w \quad \Rightarrow \quad \bar{\beta} w^2 + (\bar{\alpha}-\alpha)w - \beta = 0 \quad \Rightarrow \quad w = \frac{(\alpha-\bar{\alpha}) \pm \sqrt{(\bar{\alpha}-\alpha)^2 + 4|\beta|^2}}{2\bar{\beta}}.$$

The fixed points are determined by the discriminant $\mathfrak{D}$ of the quadratic, motivating the following set of results on the categorising of orientation-preserving Möbius transformations.

Proposition 2.30 *Let $f \in \mathcal{M}(\mathbb{D})^+$ and $\mathfrak{D}$ be the discriminant of $f(w) - w = 0$.*

(i) *If $\mathfrak{D} < 0$, then f has* **one** *fixed point in $\mathbb{D}$.*

(ii) *If $\mathfrak{D} = 0$, then f has* **one** *fixed point on $\partial\mathbb{D}$.*

(iii) *If $\mathfrak{D} > 0$, then f has* **two** *fixed points on $\partial\mathbb{D}$.*

Sketch of Proof: Consider the solutions to the quadratic equations above, case-by-case. $\square$

3 Measures of the Poincaré Disk

We now focus our discussion on objects such as length, distance and area in hyperbolic geometry and, consistent with Section 2, we will consider such measures in the Poincaré disk.

Definition 3.1 Let $I \subseteq \mathbb{R}$ be some interval. A parametrised curve in $\mathbb{R}^2$ is a smooth function $\gamma : I \to \mathbb{R}^2$; by smooth we mean that if $\gamma(t) = (x(t), y(t))$, then $x, y : I \to \mathbb{R}$ are both smooth in the usual sense. It is called a regularly parametrised curve if $\gamma'(t) \neq 0$, for all $t \in I$.

Definition 3.2 The length of a parametrised curve $\gamma : [a, b] \to \mathbb{R}^2$ is given by

$$L[\gamma] = \int_a^b \left|\gamma'(t)\right| \,\mathrm{d}t.$$

This definition is advantageous, because by selecting the interval I and curve γ in such a way that it defines a path between two points, we can manipulate Definition 3.2 to suit our needs.

3.1 Lengths in the Poincaré Disk

Recall that Definition 1.1 laid the groundwork for what the intrinsic distance is between two points in some space. This can be applied to the disk, which gives the following idea.

Definition 3.3 The Riemannian metric of the Poincaré disk is given by

$$\mathrm{d}s = \frac{2\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}}{1 - (x^2 + y^2)}.$$

Since, for our purposes, x and y are considered to be a part of the ordered pair represented by γ, parametrised by t, the formula of Definition 3.3 can be re-stated, giving the following:

$$\mathrm{d}s = \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} \,\mathrm{d}t.$$

Further, recall we noted in Section 1 that the integral of a metric gives the distance function between two points. This directly motivates the forthcoming definition.

Definition 3.4 Let $\gamma : [a, b] \to \mathbb{D}$ be a parametrised curve in the Poincaré disk having the form $\gamma(t) = (x(t), y(t))$. Then, the hyperbolic length of γ is

$$L^{\mathbb{D}}[\gamma] = \int_a^b \mathrm{d}s = \int_a^b \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} \,\mathrm{d}t.$$

As with points, it's useful to consider a parametrised curve as a complex-valued function, that is let $\gamma : I \to \mathbb{C}$, given by $\gamma(t) = x(t) + iy(t)$. Thus, γ may be referred to as a ***contour***.

We next consider the concept of a hyperbolic area. Note this will not be a major focus in Section 4 but there certainly are some interesting results regarding this concept and hyperbolic polygons (see Section 5). Nevertheless, we define the area of a region of the Poincaré disk.

Definition 3.5 The hyperbolic area of a domain $\Omega \subseteq \mathbb{D}$ given in terms of xy-coordinates is

$$A^{\mathbb{D}}[\Omega] = \iint_{\Omega} \frac{4}{(1 - x^2 - y^2)^2}\, \mathrm{d}x\mathrm{d}y.$$

3.2 Möbius Action on Length and Area

We can re-focus our discussion to situations involving Möbius transformations of the disk. We state, without proof, a lemma which reduces the proofs of Propositions 3.7 and 3.8.

Lemma 3.6 *Any Möbius transformation $f \in \mathcal{M}(\mathbb{D})$ can be written as the composition of these:*

$$\begin{aligned} r_\theta(z) &= e^{i\theta} z, && (0 \le \theta < 2\pi) \\ t_\beta(z) &= \frac{z + \beta}{\beta z + 1}, && (\beta \in \mathbb{R}) \\ s(z) &= -\bar{z}. \end{aligned}$$

Proposition 3.7 *Möbius transformations that fix the Poincaré disk are isometries of the disk.*

Proof: We must show that for every $f \in \mathcal{M}(\mathbb{D})$ and **any** parametrised curve γ, we have

$$L^{\mathbb{D}}[f \circ \gamma] = L^{\mathbb{D}}[\gamma].$$

It's sufficient to show the above is true with each of the transformations defined in Lemma 3.6. Assume that $\gamma(t) = x(t) + iy(t)$ and consider $\widetilde{\gamma}_1(t) \coloneqq t_\beta \circ \gamma \coloneqq u(t) + iv(t)$. Compute the composition directly and equate real and imaginary parts, producing the following equalities:

$$u = \frac{\beta y^2 + (x + \beta)(\beta x + 1)}{(\beta x + 1)^2 + \beta^2 y^2} \qquad \text{and} \qquad v = \frac{y - \beta^2 y}{(\beta x + 1)^2 + \beta^2 y^2}.$$

Hence, computing the squares of the above and their derivatives with respect to t gives us

$$(u')^2 = \frac{(\beta^2 - 1)^2(\beta^2 x^2 x' + 2\beta x x' + 2\beta y y' + 2\beta^2 x y y' - \beta^2 y^2 x' + x')^2}{(\beta^2 x^2 + \beta^2 y^2 + 2\beta x + 1)^4},$$

$$(v')^2 = \frac{(\beta^2 - 1)^2(\beta^2 y^2 y' - 2\beta x y' + 2\beta y x' + 2\beta^2 x y x' - \beta^2 x^2 y' - y')^2}{(\beta^2 x^2 + \beta^2 y^2 + 2\beta x + 1)^4},$$

$$(u')^2 + (v')^2 = \frac{(\beta^2-1)^2((x')^2+(y')^2)}{((\beta x+1)^2+\beta^2 y^2)^2}$$

$$1 - u^2 - v^2 = \frac{(\beta^2-1)(1-x^2-y^2)}{(\beta x+1)^2+\beta^2 y^2}.$$

Consequently, it follows, by substituting the above formulae into the integrand, that

$$\begin{aligned} L^{\mathbb{D}}[t_\beta \circ \gamma] &= \int_a^b \frac{2}{1-u^2-v^2}\sqrt{(u')^2+(v')^2}\,\mathrm{d}t \\ &= \int_a^b \frac{2}{1-x^2-y^2}\sqrt{(x')^2+(y')^2}\,\mathrm{d}t \\ &= L^{\mathbb{D}}[\gamma]. \end{aligned}$$

One can consider $\widetilde{\gamma}_2(t) \coloneqq r_\theta \circ \gamma \coloneqq p(t) + iq(t)$ and $\widetilde{\gamma}_3(t) \coloneqq s \circ \gamma \coloneqq a(t) + ib(t)$, which yield

$$p = x\cos(\theta) - y\sin(\theta), \;\; q = x\sin(\theta) + y\cos(\theta) \qquad \text{and} \qquad a = -x, \;\; b = y.$$

In either situation, it doesn't take too much work to verify the formula as for the first case. Hence, by Lemma 3.6, all Möbius transformations of the Poincaré disk can be written as the composition of such functions and, as those functions preserve the length of curves in $\mathbb{D}$, it follows that their composition, that is any such $f \in \mathcal{M}(\mathbb{D})$, will be an isometry. □

Proposition 3.8 *Möbius transformations that fix the Poincaré disk preserve area.*

Sketch of Proof: Show that each of the transformations in Lemma 3.6 preserve area. □

Example 3.9 Let $a \in \mathbb{R}$ where $0 < a < 1$. Consider the domain $\Omega = \{z \in \mathbb{D} : |z| \leq a\} \subseteq \mathbb{D}$. We can compute the area of such a circle, for arbitrary a satisfying the stated conditions. In order to apply Definition 3.5, we must find bounds on the x-variable and y-variable:

$$-\sqrt{a^2-y^2} \leq x \leq \sqrt{a^2-y^2} \qquad \text{and} \qquad -a \leq y \leq a.$$

Hence, it follows that

$$\begin{aligned} A^{\mathbb{D}}[\Omega] &= \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{4}{(1-x^2-y^2)^2}\,\mathrm{d}x\mathrm{d}y \\ &= \int_0^{2\pi}\int_0^a \frac{4r}{(1-r^2)^2}\,\mathrm{d}r\mathrm{d}\theta \end{aligned}$$

$$= \int_0^{2\pi} \frac{2a^2}{1-a^2}\,\mathrm{d}\theta$$

$$= \frac{4\pi a^2}{1-a^2}.$$

On the second line, we converted to polar coordinates where $r \in (0, a)$ and $\theta \in [0, 2\pi)$:

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta).$$

Recall also that the Jacobian of the transformation carries a factor of r, that is $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$. Thus, any disk-preserving transformation of the domain will have the same boundary length as a result of Proposition 3.7 and the same area as a consequence of Proposition 3.8.

We now develop theory to show that Möbius transformations preserve more than just length.

Definition 3.10 Let $\gamma_1 : I \to \mathbb{R}^2$ and $\gamma_2 : J \to \mathbb{R}^2$ be two regularly parametrised curves such that $\gamma_1(t) = \gamma_2(s) = p \in \mathbb{R}^2$, for some $t \in I$ and $s \in J$. The angle between γ_1 and γ_2 at p is the unique value $\theta \in [0, 2\pi)$ such that

$$\frac{\gamma_1'(t) \cdot \gamma_2'(s)}{|\gamma_1'(t)||\gamma_2'(s)|} = \cos(\theta).$$

Definition 3.11 Let $U, V \subseteq \mathbb{C}$ be open. The map $f : U \to V$ is conformal at $p \in U$ if, for regularly parametrised curves $\gamma_1 : I \to \mathbb{C}$ and $\gamma_2 : J \to \mathbb{C}$ passing through p in the sense of Definition 3.10, *the map f preserves the angles between γ_1 and γ_2 at p. Such a map is* conformal if it's conformal at all $p \in U$.

Next, we introduce ideas not uncommon when discussing complex analysis, which will inform the stating and proving of one more result regarding Möbius transformations. We note that $\mathbb{C}$ is a metric space, with a metric given by the modulus of the difference between two complex numbers, meaning Definition 1.4 applies.

Definition 3.12 For $U \subseteq \mathbb{C}$ open, we call $f : U \to \mathbb{C}$ complex differentiable at $p \in U$ if the limit

$$f'(p) := \lim_{h \to 0}\left[\frac{f(p+h) - f(p)}{h}\right]$$

exists. Such a function is holomorphic if the limit exists for all $p \in U$.

In practice, Definition 3.12 can be tedious to use directly, especially if we wish to develop a complex analytic method to verify that a function preserves angles. Therefore, it is more beneficial to appeal to the objects defined as follows.

Definition 3.13 Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ where $f(z) = u(x, y) + iv(x, y)$ and $z = x + iy$.

Then, f is holomorphic if and only if the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Despite this, it is still difficult to show that a Möbius transformation is holomorphic on the Riemann sphere since the separation into real and imaginary parts is, to say the least, messy. Hence, we appeal to an idea discussed in [Nar00], whose proof is beyond the discussion here.

Theorem 3.14 (Looman–Menchoff Theorem) *Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ continuous. Suppose further that the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at every $z = x + iy \in U$, satisfying*

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

Then, f is holomorphic on U.

Lemma 3.15 *The Möbius transformations $\mathcal{M}(\mathbb{C} \setminus \{-d/c\})$ are holomorphic.*

Proof: Consider f as in Definition 2.11, where now $z \in \mathbb{C} \setminus \{-d/c\}$. For $z = x + iy$, we get that

$$\frac{\partial f}{\partial x} = \frac{ad - bc}{(c(x + iy) + d)^2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{i(ad - bc)}{(c(x + iy) + d)^2}.$$

Therefore, we can verify one of the conditions of Theorem 3.14, namely

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0.$$

We verify that $\mathbb{C} \setminus \{-d/c\} \subseteq \mathbb{C}$ is open: consider the element $w \in \mathbb{C} \setminus \{-d/c\}$ and define $\varepsilon = |w + d/c| > 0$. Then, $B_\varepsilon(w) \subseteq \mathbb{C} \setminus \{-d/c\}$ and Definition 1.4 applies. Finally, f is continuous as shown by the stronger result in [Duc06]. Therefore, the Looman-Menchoff Theorem applies, giving the result. □

Lemma 3.16 (Conformal Mapping Theorem) *Suppose $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is some function holomorphic at $p \in U$. If $f'(p) \neq 0$, then f is conformal at p.*

Proof: Let $\gamma_1 : I \to \mathbb{C}$ and $\gamma_2 : J \to \mathbb{C}$ be regular curves as in Definition 3.1 and suppose that $\gamma_1(t_1) = p = \gamma_2(t_2)$, for $t_1 \in I$ and $t_2 \in J$. By Definition 3.10, the angle between the curves at p is the unique value $\theta \in [0, 2\pi)$, satisfying

$$\frac{\gamma_1'(t_1) \cdot \gamma_2'(t_2)}{|\gamma_1'(t_1)||\gamma_2'(t_2)|} = \cos(\theta).$$

The derivatives of $f \circ \gamma_1$ and $f \circ \gamma_2$ follow from the Chain Rule of differentiation, giving

$$\frac{f'(p)\gamma_1'(t_1) \cdot f'(p)\gamma_2'(t_2)}{f'(p)|\gamma_1'(t_1)|f'(p)|\gamma_2'(t_2)|} = \frac{\gamma_1'(t_1) \cdot \gamma_2'(t_2)}{|\gamma_1'(t_1)||\gamma_2'(t_2)|} = \cos(\theta).$$

Thus, the angle between the curves is invariant under f, that is f is conformal at p. □

One of the celebrated properties of Möbius transformations is that they are conformal maps as functions defined on the Riemann sphere. We are now equipped to prove this.

Theorem 3.17 *The set $\mathcal{M}(\overline{\mathbb{C}})$ consists of conformal maps from and to the Riemann sphere.*

Proof: Let $f \in \mathcal{M}(\overline{\mathbb{C}})$. We deepen the result as discussed in [Bul09] by considering two cases.

(i) $c \neq 0$: In this case, the Möbius transformations concerning the assumption are given as

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where } ad - bc \neq 0 \text{ and } a, b, c, d \in \mathbb{C}.$$

By applying the Quotient Rule of differentiation, one can compute

$$f'(z) = \frac{ad-bc}{(cz+d)^2}.$$

The only pole of $f'(z)$ is the point $z = -d/c$, meaning $f'(z) \neq 0$, for all $z \in \mathbb{C} \setminus \{-d/c\}$. Define $g(z) \coloneqq 1/f(z)$. By again applying the Quotient Rule and simplifying, we obtain

$$g'(z) = -\frac{ad-bc}{(az+b)^2}.$$

Therefore, $g'(-d/c) = -c^2/(ad-bc) \neq 0$, meaning that g is conformal at $z = -d/c$ and, as a result, that f is conformal at $z = -d/c$ also. Consider $f(1/z) = (a+bz)/(c+dz)$. The Quotient Rule yields the following result:

$$f'\left(\frac{1}{z}\right) = \frac{bc-ad}{(c+dz)^2}.$$

Therefore, $f'(1/0) \coloneqq f'(\infty) = (bc-ad)/(c^2) \neq 0$, meaning f is conformal at $z = \infty$.

(ii) $c = 0$: In this case, the Möbius transformations concerning the assumption are given as

$$f(z) = \frac{az+b}{d}, \quad \text{where } ad \neq 0 \text{ and } a, b, d \in \mathbb{C}.$$

Now, $a, d \neq 0$, otherwise the condition above is violated. It immediately follows that

$$f'(z) = \frac{a}{d} \neq 0, \quad \text{for all } z \in \mathbb{C}.$$

Consider $f(1/z) = (a + bz)/dz$. Differentiating this relation gives the following result:

$$f'\left(\frac{1}{z}\right) = -\frac{a}{dz^2}.$$

Therefore, $f'(1/0) \coloneqq f'(\infty) = \infty \neq 0$, meaning f is conformal at $z = \infty$.

Consequently, $\mathcal{M}(\overline{\mathbb{C}})$ consists of conformal maps, by the Conformal Mapping Theorem. $\square$

Remark 3.18 We can differentiate the transformation in the proof of Theorem 3.17 as a result of Lemma 3.15. This is because we realise that f is meromorphic on $\mathbb{C}$, that is holomorphic at all but finitely many points of its domain; we rectify any issues of holomorphism over the complex plane by the fact that the transformation is defined also at the point at infinity.

Corollary 3.19 *The set $\mathcal{M}(\mathbb{D})$ consists of conformal maps from and to the Poincaré disk.*

Proof: This is a direct consequence of Proposition 2.22 and Theorem 3.17. $\square$

Here ends the discussion on introductory hyperbolic geometry. By no means is this a complete dissection of the subject but this provides a solid ground upon which to begin the study of hyperbolic polygons with right-angles.

4 Hyperbolic Polygons with Right-Angles

Elementary properties of hyperbolic geometry have been discussed and proved; we now begin to explore some right-angled shapes, specifically in the Poincaré disk.

4.1 Hyperbolic Triangles and Trigonometry

The first shape to be discussed is the triangle. The idea of such a polygon is broadly the same as in the Euclidean definition, but we will soon see some fundamental differences.

Definition 4.1 A hyperbolic triangle in $\mathbb{D}$ is the set of three points $A, B, C \in \mathbb{D}$ such that they are non-collinear and are connected by hyperbolic line segments.

We now specialise in the case where a hyperbolic triangle has (at least) one right-angle. It will become clear that the related trigonometric identities for such polygons are specific to this case, but first we recall the definitions of sinh, cosh and tanh.

Definition 4.2 The hyperbolic sine, hyperbolic cosine and hyperbolic tangent functions are defined as follows, for $z \in \mathbb{C}$:

$$\sinh(z) = \frac{e^z - e^{-z}}{2}, \qquad \cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

Now, Definition 4.2 is a needed starting point for the development of hyperbolic trigonometric identities. Before we state and prove such results, we must first define a number of objects which establish key ideas that we can exploit.

Definition 4.3 A radical axis of two circles is the locus of points from which tangent lines of equal length can be drawn to each of the circles. In particular, for two non-disjoint circles, the radical axis is the Euclidean line which contains the intersecting points.

Definition 4.4 Let $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$ be distinct points. Then, their cross-ratio is defined as

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

Example 4.5 Consider the following cross-ratio, for arbitrary $z \in \mathbb{D}$:

$$[z, 0; 1, -1] = \frac{(z-1)(0+1)}{(z+1)(0-1)} = \frac{z-1}{-z+1}.$$

This satisfies Proposition 2.14, meaning it's a Möbius transformation of the Poincaré disk.

Remark 4.6 The cross-ratio can actually be used to construct a Möbius transformation which maps specified points to other specified points. The way in which to find the conformal map

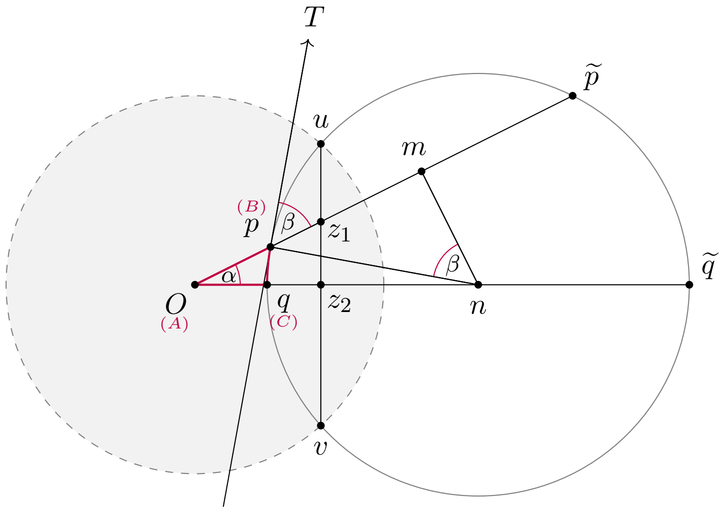

Figure 4: Right-angled triangle as in Theorem 4.10 and Corollary 4.11, like [Jab08, Figure 4].

that sends z_1, z_2, z_3 to w_1, w_2, w_3 is to solve $[z, z_1; z_2, z_3] = [f(z), w_1; w_2, w_3]$ for $f(z)$.

Before we can state and prove some trigonometric identities of right-angled hyperbolic triangles, we introduce a final notion of distance. This will be a reformulation of the idea of distance seen in Definition 3.4.

Definition 4.7 Let $a, b \in \mathbb{D}$ be distinct. The Poincaré length between a and b is defined as

$$\rho(a, b) = \log([a, b; p, q]),$$

for $p, q \in \partial\mathbb{D}$ the endpoints of the hyperbolic line segment through a and b, where log is the natural logarithm.

Remark 4.8 It is known, by [Con17], that the cross-ratio of points that lie on a Euclidean circle or line, in particular a hyperbolic line in the Poincaré disk, is real. Thus, the formula in Definition 4.7 does make sense since Proposition 2.8 gives that a and b are on some such line.

Example 4.9 This is inspired by (but elaborates upon) [Hay08]; let O be the origin and suppose $p \in \mathbb{D}$ and $u, v \in \partial\mathbb{D}$ lie on a diameter of the Poincaré disk. Without loss of generality, assume that said diameter is precisely the intersection of the disk with the real axis and that, from left-to-right, the points are u, O, p, v. Hence, it follows that

$$\rho(O, p) = \log([O, p; u, v]) = \log\left(\frac{1 + |p|}{1 - |p|}\right),$$

since, by definition, $|u| = |v| = 1$, where we also use $|u - p| = 1 + |p|$ and $|v - p| = 1 - |p|$. Thus, as this is a real number, we can exponentiate, resulting in the following equality:

$$e^{\rho(O,p)} = \frac{1+|p|}{1-|p|}.$$

As a near-immediate consequence of Definition 4.2, therefore, we get these expressions:

$$\sinh(\rho(O,p)) = \frac{2|p|}{1-|p|^2}, \qquad \cosh(\rho(O,p)) = \frac{1+|p|^2}{1-|p|^2}, \qquad \tanh(\rho(O,p)) = \frac{2|p|}{1+|p|^2}.$$

Note that we will abuse notation *slightly* in the proof of the following theorem by referring to the modulus of the difference of points in the plane as the Euclidean line joining them. Be aware that this is indeed just notation alone.

Theorem 4.10 *Suppose ABC is a right-angled hyperbolic triangle with A the origin and C the right-angle. Assume further that the* **hyperbolic** *lengths of the edges BC, AC, AB are a, b, c and angles at A, B, C labelled α, β, γ, respectively, Consequently, it follows that*

$$\sin(\alpha) = \frac{\sinh(a)}{\sinh(c)} \qquad \textit{and} \qquad \cos(\alpha) = \frac{\tanh(b)}{\tanh(c)}.$$

Proof: Let ABC be as in Figure 4. Definition 2.3 gives that the hyperbolic line joining p and q is the arc of some circle C, centred at n, say. By construction, C contains the inverse points $\tilde{p}$ and $\tilde{q}$. Define u and v to be the intersections of C and $\partial\mathbb{D}$; the line segment connecting these points is the radical axis, which intersects the line segments $|\tilde{p}|$ and $|\tilde{q}|$ at z_1 and z_2, respectively. So,

$$|\tilde{p} - p| = \frac{1-|p|^2}{|p|} = \frac{2}{\sinh(c)} \qquad \text{and} \qquad |\tilde{q} - q| = \frac{1-|q|^2}{|q|} = \frac{2}{\sinh(b)},$$

where equality in terms of the hyperbolic sine function comes from Example 4.9. Now, note that m is defined to be the point such that $|p-m| = \frac{1}{2}|\tilde{p}-p|$. and T is the line tangent to C at the point p. So, by Euclidean trigonometry applied to the triangle with vertices p, m, n,

$$\sin(\beta) = \frac{|p-m|}{|q-n|} = \frac{\frac{1}{2}|\tilde{p}-p|}{\frac{1}{2}|\tilde{q}-q|} = \frac{\sinh(b)}{\sinh(c)}.$$

The conclusion is that the sine of an angle is the quotient of the hyperbolic sines of the lengths of the side opposite the angle and the side opposite the right-angle. Hence, applying this then to angle α gives the intended result. Next, [Joh60] provides the following useful formula:

$$|z_1|^2 - 1 = |p - z_1||\tilde{p} - z_1| = |q - z_2||\tilde{q} - z_2|.$$

Consequently, taking each of these equalities generates the following argument:

$$\begin{aligned} |z_1|^2 - 1 &= |p - z_1||\tilde{p} - z_1| \\ &= 1 - |z_1||p| - |z_1|/|p| + |z_1|^2 \\ \Rightarrow \qquad |z_1| &= \frac{2|p|}{1+|p|^2} \\ &= \tanh(c) \end{aligned}$$

again by Example 4.9. By identical reasoning to that just exhibited, we also conclude that

$$\begin{aligned} |z_2| &= \frac{2|q|}{1+|q|^2} \\ &= \tanh(b). \end{aligned}$$

Hence, by Euclidean trigonometry applied to the triangle with vertices O, z_1, z_2, we see that

$$\cos(\alpha) = \frac{|z_2|}{|z_1|} = \frac{\tanh(b)}{\tanh(c)}. \qquad \square$$

As mentioned, the proof of Theorem 4.10 uses Figure 4 as a model. Note it applies to any such hyperbolic right-angled triangle ABC, with C being the right-angle, since any such triangle can be 'moved' so that it has a vertex at the origin, that is apply a Möbius transformation, which is now known to both be an isometry of the Poincaré disk and to preserve angles. These facts were established in Proposition 3.7 and Theorem 3.17.

Corollary 4.11 *Suppose ABC is a right-angled hyperbolic triangle with A the origin and C the right-angle. Assume further that the* **hyperbolic** *lengths of the edges BC, AC, AB are a, b, c and angles at A, B, C labelled α, β, γ, respectively, Consequently, it follows that*

$$\cosh(a)\cosh(b) = \cosh(c) \qquad \textit{and} \qquad \tan(\alpha) = \frac{\tanh(a)}{\sinh(b)}.$$

Proof: First, we prove the cosh identity by appealing to Theorem 4.10. This is inspired by but distinct from [Val06]; note it is sufficient to prove the square of the identity. Now, we have that

$$\begin{aligned} \sin^2(\alpha) + \cos^2(\alpha) &= 1 \\ \Rightarrow \qquad \sinh^2(a) + \tanh^2(b)\cosh^2(c) &= \sinh^2(c) \\ \Rightarrow \qquad 1 + \sinh^2(a) + \tanh^2(b)\cosh^2(c) &= 1 + \sinh^2(c) \\ \Rightarrow \qquad \cosh^2(a)\cosh^2(b) + \sinh^2(b)\cosh^2(c) &= \cosh^2(c)\cosh^2(b) \\ \Rightarrow \qquad \cosh^2(a)\cosh^2(b) &= \cosh^2(c)(\cosh^2(b) - \sinh^2(b)) \end{aligned}$$

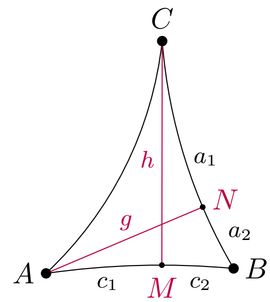

Figure 5: Triangle ABC as in Theorems 4.13 and 4.14.

$$= \cosh^2(c)$$

Furthermore, we can apply the above result to obtain the last identity, that is

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{\sinh(a)\tanh(c)}{\sinh(c)\tanh(b)} = \frac{\sinh(a)}{\cosh(a)\cosh(b)\tanh(b)} = \frac{\tanh(a)}{\sinh(b)}. \qquad \square$$

Now that expressions for the sine, cosine and tangent of the non-right-angles in the right-angled triangle are proved, as well as one relating each of the sides of a hyperbolic triangle (which is analogous to Pythagoras' Theorem), we can state versions of the sine and cosine rules.

Remark 4.12 In the sequel, we exclusively refer to the vertices of the polygons as A, B, C and so forth, with the angle at each vertex the corresponding Greek letter, unless otherwise stated. Specifically, these are points in the Poincaré disk but Theorem 4.10 and Corollary 4.11 are suitable to extend the theory without the need of a rigorous construction as in Figure 4. This allows us to relax notation: $ABC...$ will represent the polygon.

Theorem 4.13 (Hyperbolic Sine Rule) *Suppose that ABC is a hyperbolic triangle, labelled as in* Figure 5. *Then, it follows that*

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}.$$

Proof: First of all, note that ABC the union of the two triangles ACM and BCM. Then, by initially considering triangle ACM, we have that

$$\sin(\alpha) = \frac{\sinh(h)}{\sinh(b)} \qquad \Rightarrow \qquad \sinh(h) = \sin(\alpha)\sinh(b),$$

where we appeal to Theorem 4.10. In a similar vein, for the triangle BCM, we have that

$$\sin(\beta) = \frac{\sinh(h)}{\sinh(a)} \qquad \Rightarrow \qquad \sinh(h) = \sin(\beta)\sinh(a).$$

Equating the above and re-arranging gives the first equality. Likewise, we can consider ABC the union of the triangles ACN and ABN, showing the second equality. □

Theorem 4.14 (Hyperbolic Cosine Rule 1) *Suppose that ABC is a hyperbolic triangle, labelled as in* Figure 5. *Then, it follows that*

$$\cosh(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(\alpha).$$

Proof: We apply Corollary 4.11 to the triangle BCM:

$$\begin{aligned}
\cosh(a) &= \cosh(h)\cosh(c_2) \\
&= \cosh(h)\cosh(c - c_1) \\
&= \cosh(h)(\cosh(c)\cosh(c_1) - \sinh(c)\sinh(c_1)) && \textbf{(4.15)} \\
&= \cosh(c)\cosh(c_1)\cosh(h) - \sinh(c)\sinh(c_1)\cosh(h) \\
&= \cosh(b)\cosh(c) - \sinh(c)\cosh(b)\tanh(c_1) && \textbf{(4.16)} \\
&= \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(\alpha),
\end{aligned}$$

where (4.15) follows from applying the identity $\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$ and where (4.16) follows from Corollary 4.11 applied to the triangle ACM. □

Corollary 4.17 (Hyperbolic Cosine Rule 2) *Suppose that ABC is a hyperbolic triangle, labelled as in* Figure 5. *Then, it follows that*

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

Proof: This follows from manipulations of the results of Theorems 4.10 and 4.14. □

Remark 4.18 The second cosine rule for hyperbolic triangles does *not* have an analogous relation in Euclidean space; consider now what the result states about hyperbolic triangles: *the side length of one of its sides is uniquely determined by the size of its angles.* This is false in Euclidean geometry; triangles with congruent angles aren't congruent generally.

4.2 Lambert Quadrilaterals

We will now study a special type of quadrilateral in hyperbolic space, whose identities follow from the previous work on hyperbolic triangles.

Definition 4.19 A hyperbolic quadrilateral in $\mathbb{D}$ is the set of points $A, B, C, D \in \mathbb{D}$ such that no three are collinear, consisting of four hyperbolic line segments connected cyclically.

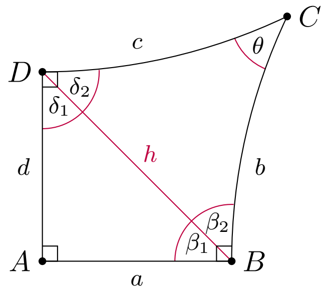

Figure 6: Lambert quadrilateral $ABCD$ as in Propositions 4.24, 4.25 and Corollary 4.26.

Definition 4.20 Two hyperbolic lines are called parallel if they do not intersect anywhere other than at the boundary of the Poincaré disk. If they are completely disjoint, then they are said to be ultra-parallel.

We first look at a related but, for our purposes, less significant polygon in order to develop results for the special type of quadrilateral this section is dedicated to.

Definition 4.21 A Saccheri quadrilateral $ABCD$ is a quadrilateral such that the sides AD, BC are parallel, of equal length, with right-angles at the base angles α and β.

Proposition 4.22 *Let $ABCD$ be a Saccheri quadrilateral, labelled anti-clockwise with base length a, vertical length b and summit (i.e. upper edge) length c. Then, the summit angles are equal and the summit CD is perpendicular to the perpendicular bisector of AB. Furthermore, the following formula relates the lengths of each of the sides of the shape:*

$$\sinh\left(\frac{c}{2}\right) = \sinh\left(\frac{a}{2}\right)\cosh(b).$$

Proof: See [Sib15, Gre03] for each of the properties. □

We now prove (some) formulae for the titular polygon. This is important not only in its own right but for the development of theory regarding the final polygon studied in this paper.

Definition 4.23 A Lambert quadrilateral is a quadrilateral with at least three right-angles.

Proposition 4.24 *Let $ABCD$ be a Lambert quadrilateral, with acute angle θ at C. Then, given the labelling as in* Figure 6, *it follows that*

$$\sin(\theta) = \frac{\cosh(a)}{\cosh(c)}.$$

Sketch of Proof: This follows by first applying Theorem 4.10 to the triangle BCD. The known identity $\sin(x) = \cos(\pi/2 - x)$ and Corollary 4.11 are then utilised, before appealing to Proposition 4.22, treating b as the summit, d the base and c the vertical height. Note for a Lambert quadrilateral, the side lengths are half those of a Saccheri quadrilateral, meaning that we have the equality $\sinh(b) = \sinh(d)\cosh(c)$. Combining these results in sequence gives us a sufficient way to complete the proof. $\square$

Proposition 4.25 *Let $ABCD$ be a Lambert quadrilateral, with acute angle θ at C. Then, given the labelling as in* Figure 6*, it follows that*

$$\cos(\theta) = \sinh(a)\sinh(d).$$

Proof: Here, we apply the hyperbolic cosine rule (second version) to triangle BCD, giving

$$\cos(\theta) = \cosh(h)\sin(\beta_2)\sin(\delta_2) - \cos(\beta_2)\cos(\delta_2).$$

Now then, the formula $\sin(x) = \cos(\pi/2 - x)$ can once again be applied:

$$\begin{aligned}\cos(\theta) &= \cosh(a)\cosh(d)\cos(\beta_1)\cos(\delta_1) - \sin(\beta_1)\sin(\delta_1)\\ &= \cosh(a)\cosh(d)\tanh(a)\tanh(d)\coth^2(h) - \sinh(a)\sinh(d)\operatorname{csch}^2(h)\\ &= \sinh(a)\sinh(d)(\coth^2(h) - \operatorname{csch}^2(h))\\ &= \sinh(a)\sinh(d),\end{aligned}$$

again appealing to Theorem 4.10, Corollary 4.11 and using known trigonometric identities. $\square$

Corollary 4.26 *Let $ABCD$ be a Lambert quadrilateral, with acute angle θ at C. Then, given the labelling as in* Figure 6*, it follows that*

$$\cot(\theta) = \sinh(b)\tanh(a).$$

Sketch of Proof: The result follows by expressing $\cot(\theta)$ as a quotient of cosine and sine and applying a version of the formula seen in Proposition 4.22. $\square$

This concludes the look into Lambert quadrilaterals and naturally leads into the study of a five-sided polygon with right-angles.

4.3 Hyperbolic Pentagons

The final area for discussion will be that regarding hyperbolic right-angled pentagons. The first set of results give formulae relating the different side lengths of the pentagon, proved here using trigonometric identities; note that other texts often appeal to different methods.

Definition 4.27 A hyperbolic right-angled pentagon in $\mathbb{D}$ is the set of points $A, B, C, D, E \in \mathbb{D}$ such that no three are collinear, consisting of five hyperbolic line segments connected cyclically,

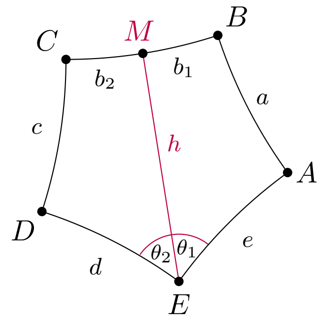

Figure 7: Right-angled pentagon $ABCDE$ as in Propositions 4.28, 4.31 and Corollary 4.29.

intersecting at right-angles only at the aforementioned vertices.

Proposition 4.28 *Let $ABCDE$ be a right-angled pentagon, labelled as in* Figure 7. *Then,*

$$\cosh(a)\cosh(c) = \sinh(a)\sinh(c)\cosh(b).$$

Sketch of Proof: One can use [Bea83] for a general case where the hyperbolic pentagon has four right-angles and one acute angle, in which the result follows by substituting in $\pi/2$ for the acute angle. In essence, use the fact that $ABME$ and $DCME$ are Lambert quadrilaterals and appeal to Propositions 4.24 and 4.25. This allows for one to subtract zero from $\cosh(a)\cosh(c)$ in a non-trivial way, leading to the result. □

It is important to note that, as with other polygons studied, the sides/vertices are arbitrarily chosen in the statements of these results; Proposition 4.28 is no different. Therefore, we can interpret the above as follows: *the product of the* cosh *of two non-adjacent sides of a hyperbolic right-angled pentagon is equal to the product of the* sinh *of those sides and the* cosh *of the side adjacent to both*. This fact will be used to prove that which follows.

Corollary 4.29 *Let $ABCDE$ be a right-angled pentagon, labelled as in* Figure 7. *Then,*

$$\tanh(a)\tanh(c)\cosh(b) = 1 \qquad \textit{and} \qquad \cosh(d) = \sinh(a)\sinh(b).$$

Proof: The first result follows directly from Proposition 4.28; divide through by $\cosh(a)\cosh(c)$ to get the desired formula. The second result is more involved. One can apply Proposition 4.28 to sides c and e, giving

$$\cosh(c)\cosh(e) = \sinh(c)\sinh(e)\cosh(d).$$

Therefore, we can isolate an expression for the side length c, leading to this expression:

$$\tanh(c) = \frac{\cosh(e)}{\sinh(c)\cosh(d)}.$$

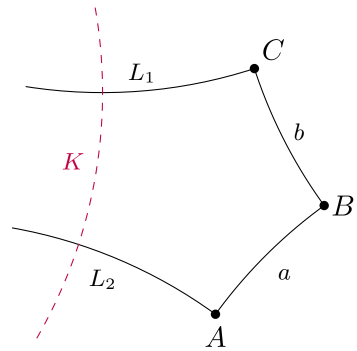

Figure 8: Constructing a right-angled pentagon as in Proposition 4.31.

This can be substituted into the first result of this corollary and manipulated, which gives

$$\tanh(a)\cosh(b)\cosh(e) = \sinh(e)\cosh(d) \qquad \Rightarrow \qquad \cosh(d) = \tanh(a)\cosh(b)\coth(e).$$

Now, applying the first result of this corollary to the sides e, a and b, we get

$$\tanh(e)\cosh(a)\tanh(b) = 1 \qquad \Rightarrow \qquad \coth(e) = \cosh(a)\tanh(b).$$

Finally then, the above relations immediately imply the second desired expression, that is

$$\begin{aligned}\cosh(d) &= \tanh(a)\cosh(b)\cosh(a)\tanh(b)\\ &= \sinh(a)\sinh(b).\end{aligned}$$

Thus, the desired result is achieved. □

Finally, we conclude the discussion on hyperbolic pentagons with a necessary and sufficient condition which allows for the construction of such a shape with given adjacent side lengths. We first state, without proof, a result which will be used to prove Proposition 4.31.

Lemma 4.30 *Two hyperbolic lines are ultra-parallel if and only if there exists a third hyperbolic line which intersects each of them perpendicularly. In particular, if the lines intersect, then there exists no such perpendicular third line.*

Proposition 4.31 *Consider some right-angled pentagon $ABCDE$. Then, it has adjacent sides of lengths $a, b \in \mathbb{R}$ if and only if $\sinh(a)\sinh(b) > 1$.*

Proof: ($\Rightarrow$) Trivial, since Corollary 4.29 gives that $\sinh(a)\sinh(b) = \cosh(d) \geq 1$, with equality if and only if $d = 0$, but $d \neq 0$ since it is the length of the edge of a polygon.

($\Leftarrow$) Suppose $\sinh(a)\sinh(b) > 1$. We construct hyperbolic line segments connecting the points

A, B, C, as in Figure 8, such that the hyperbolic distance between A and B is a; the hyperbolic distance between B and C is b and the segments form a right-angle at B. Then, consider the hyperbolic lines L_1 and L_2 extending from A and C, respectively, such that they form right-angles at the aforementioned points. But now, these lines intersect another hyperbolic line, K say, perpendicularly. Indeed, if L_1 and L_2 intersect within the Poincaré disk, at $D \in \mathbb{D}$ say, this means $ABCD$ is a Lambert quadrilateral. Hence the acute angle θ, at D, satisfies $\sinh(a)\sinh(b) = \cos(\theta)$, by Proposition 4.25, but this contradicts $\sinh(a)\sinh(b) > 1$. Hence, it is clear that L_1 and L_2 don't intersect inside the disk. As a result, they are ultra-parallel and Lemma 4.30 applies. Consequently, there exists some hyperbolic line, K, intersecting L_1 and L_2 perpendicularly at some points $D, E \in \mathbb{D}$. Thus, the conditions of Definition 4.27 are satisfied by $ABCDE$. $\square$

Remark 4.32 It is possible to form an algorithm which can be used to construct regular right-angled hyperbolic n-gons; the statement of Proposition 4.31, that it is possible to construct an irregular pentagon with specified adjacent lengths, isn't completely obvious at first glance.

The discussion on right-angled hyperbolic polygons now comes to an end. Now then, as with Sections 2 and 3, there is much more left to be discussed, a point elaborated upon in the subsequent section.

5 Discussion

This concludes the excursion into hyperbolic geometry in a general sense and, more specifically, in looking at properties of some particular polygons. The groundwork established initially by discussing rudimentary hyperbolic geometry was sufficient to allow for the expansion into the area of right-angled polygons. This itself was developed in such a way to allow for the later work and earlier results to complement each another.

5.1 Extending the Discussion

There are a number of possibilities as to where the discussion could go, as considered below. To begin with, a natural continuation would be to study right-angled hyperbolic hexagons. In fact, a satisfying property exhibited is that of the sine rule, as adapted from [Thu02].

Theorem (Sine Rule for Hyperbolic Hexagons) *Let $ABCDEF$ be a right-angled hyperbolic hexagon with sides $(a, d), (b, e), (c, f)$ pairwise opposite. Then,*

$$\frac{\sinh(a)}{\sinh(d)} = \frac{\sinh(b)}{\sinh(e)} = \frac{\sinh(c)}{\sinh(f)}.$$

Better yet, one of the most natural routes to take now would be to look at n-gons in general. More specifically, we could introduce technical properties of hyperbolic space (e.g. curvature), leading to the Gauss-Bonnet Theorem for hyperbolic triangles.

Theorem (Gauss-Bonnet Theorem for Hyperbolic Triangles) *Let ABC be some hyperbolic triangle with angles α, β, γ at A, B, C, respectively, and an enclosed region Δ. Consequently,*

$$A^{\mathbb{D}}[\Delta] = \pi - (\alpha + \beta + \gamma).$$

The area of any hyperbolic polygon follows inductively, without need of Definition 3.5; it would be interesting to state properties of the area of a general n-gon and then to research some relationships between lengths of adjacent/non-adjacent sides.

One could even consider other models of hyperbolic geometry, or even higher-dimensional extensions to the ones covered in this paper. This in itself could be a large undertaking, if we not only discuss properties of points, lines and transformations in other models but construct isomorphisms to show precisely that each model indeed does represent the same geometry.

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